Coil Shape Gradients for Island Width Minimization in Stellarators

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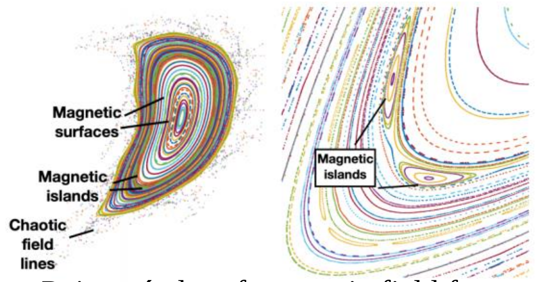
Introduction and Motivation

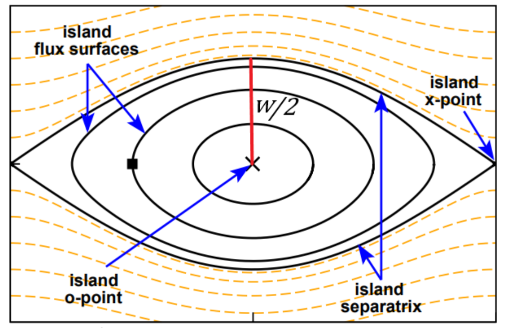
Magnetic Islands are Inherent to Stellarator Equilibria

- 3D equilibria are non-integrable in general
- Magnetic islands affect confinement and stability motivating this work on island elimination



Poincaré plot of magnetic field from example NCSX coil set [1]

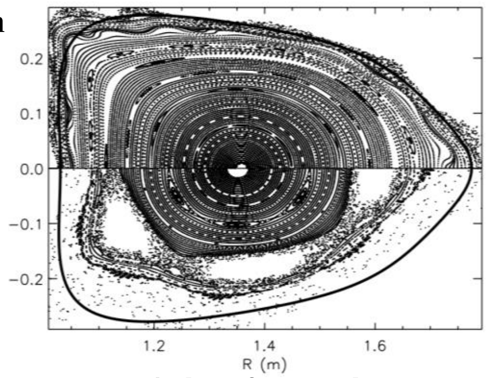
- Magnetic islands are bound by a separatrix which defines the island width, w and
- Magnetic islands have two periodic field lines, called the O and X points



Flux surfaces plotted on Poincaré section with important island features labeled

Previous Island Optimization in Stellarators

- Magnetic islands can be optimized for in many ways
  - Greene's residue
    - Not directly related to island width
    - Used for CTH
  - Resonant Fourier Harmonic
    - Is directly related to island width
    - Requires quadratic flux minimizing surfaces
    - Used for NCSX
  - Total reconnected helical flux (helical flux)
    - Is directly related to island width
    - Only requires periodic magnetic field lines
    - Used for this work



Poincaré plot of original NCSX magnetic field on bottom and "healed" magnetic field on top [2]

Shape Gradients and Sensitivity

- NCSX islands are extremely sensitive thus making coil tolerances stringent and motivating minimization of the island width sensitivity
- Island width sensitivity has been diagnosed with Eigenvalues of the Hessian matrix for the resonant Fourier harmonic
  - Requires second order variations
  - Difficult to target in optimization
- Coil shape gradients for island width can be used to diagnose island width sensitivity
  - Only requires first order variations

Shape gradient for drag on Corvette. Outwards normal perturbations of red regions increase drag and blue regions decrease drag

Island Width Objective Function

Magnetic Field with Closed Flux Surfaces is Perturbed

- Consider a magnetic field $\mathbf{B_0}$ that is comprised of a series of closed flux surfaces and given in its canonical form

$$\mathbf{B_0} = \nabla\psi_0 \times \nabla\theta + \iota\nabla\varphi \times \nabla\psi_0$$

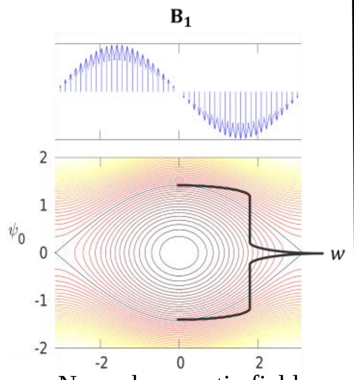
- This magnetic field is perturbed (varied) by $\mathbf{B_1}$ and the total field is now $\mathbf{B} = \mathbf{B_0} + \mathbf{B_1}$
- The helical flux, ψ_{hel} is the magnetic flux through a ribbon connecting the O points to the X points

$$\psi_{hel} = -\frac{2}{\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\mathbf{B}\cdot\nabla\psi_0}{\mathbf{B}_0\cdot\nabla\varphi}\sin(m_r\theta - n_r\varphi)\,d\theta\,d\varphi$$

poloidal and toroidal resonant mode numbers

$$w = 2\sqrt{\frac{|\psi_{hel}|}{\pi m_r \iota_r'}}$$

global shear at rational surface $\iota_r = n_r/m_r$



Normal magnetic field perturbation and resulting island in slab geometry

Island Width Objective Function

- Island width variations are inversely proportional to island width and go to infinity as the island width goes to zero

$$f_w = \psi_{hel}^2$$
$$\delta f_w = 2\psi_{hel}\delta\psi_{hel}$$

- f_w is an island width objective function that has a global minimum when the island width is zero and variations of f_w are well behaved
- Using Stokes' theorem the helical flux can be written as the difference in the magnetic action of the periodic field lines

$$\psi_{hel} = \oint \mathbf{A}\cdot d\mathbf{l}_O - \oint \mathbf{A}\cdot d\mathbf{l}_X$$

- Variations of the helical flux assume the locations of the periodic field lines stay fixed

$$\delta\psi_{hel} = \oint \delta\mathbf{A}\cdot d\mathbf{l}_O - \oint \delta\mathbf{A}\cdot d\mathbf{l}_X$$

Coil Shape Gradient and Tolerance Functional

- The magnetic field is enforced to come from a series of single-filament coils (vacuum fields)

$$\mathbf{A} = \sum_{i=1}^{N_c}\frac{\mu_0 I_i}{4\pi}\int_0^{2\pi}\frac{\mathbf{r}_i'}{|\mathbf{r}-\mathbf{r}_i|}\,d\zeta$$

i^{th} filamentary coil

- Variations of the magnetic field are assumed to be linear and are given as

variation of i^{th} coil

$$\delta\mathbf{A} = \int_0^{2\pi}\sum_{i=1}^{N_c}\left(\frac{\delta\mathbf{A}}{\delta\mathbf{r}_i}\right)^T\delta\mathbf{r}_i\,d\zeta \qquad \left(\frac{\delta\mathbf{A}}{\delta\mathbf{r}_i}\right)^T = \frac{\mu_0 I_i}{4\pi}\left(\frac{\mathbf{r}_i'(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} - \frac{(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{r}_i'}{|\mathbf{r}-\mathbf{r}_i|^3}\mathbf{I}\right)$$

- Variations of the helical flux, also referred to as coil shape gradients for the helical flux

$$\frac{\delta\psi_{hel}}{\delta\mathbf{r}_i} = \oint\frac{\delta\mathbf{A}}{\delta\mathbf{r}_i}d\mathbf{l}_O - \oint\frac{\delta\mathbf{A}}{\delta\mathbf{r}_i}d\mathbf{l}_X$$

- A coil tolerance functional is given that depends on the coil shape gradient for the helical flux

$$T_\psi = \frac{\Delta\psi_{hel}}{\int_0^{2\pi}\sum_{i=1}^{N_c}\left|\frac{\delta\psi_{hel}}{\delta\mathbf{r}_i}\right|d\zeta}$$

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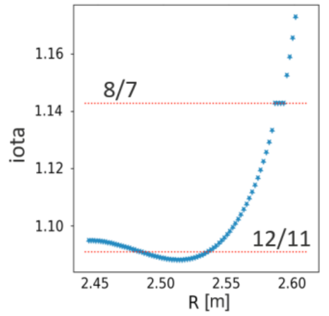
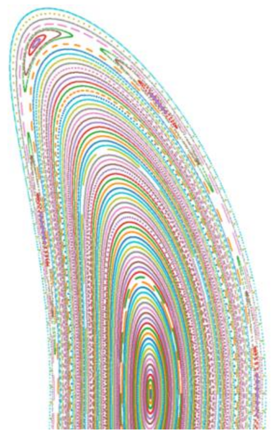
WISTELL-A Island Width Minimization Results

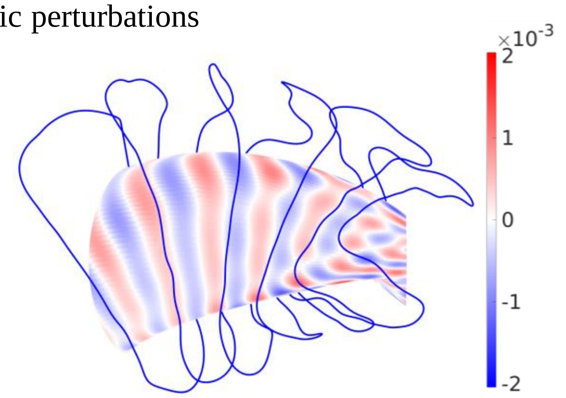
- The WISTELL-A configuration is a QHS stellarator [3]
  - Good energetic particle confinement
  - Good quasi-symmetry
  - 4 field periods
  - Vacuum

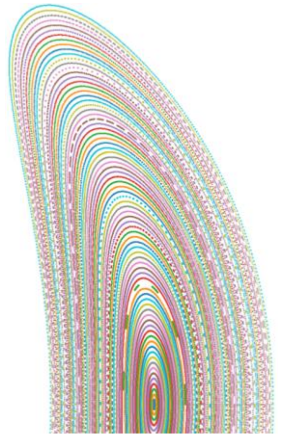




Original fixed-boundary

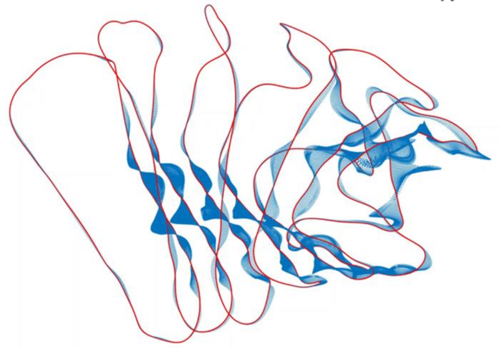
- Poincaré plot from original fixed-boundary equilibrium shows sizeable 8/7 island
- Boundary optimized with SIMSOPT to eliminate island by targeting Greene's residue
- Very small changes to the plasma boundary eliminated the 8/7 island indicating the island's high sensitivity to magnetic perturbations

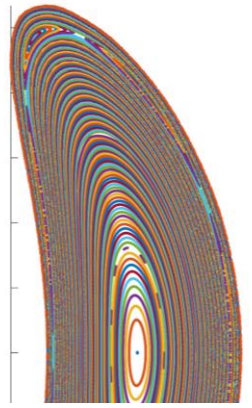




Fixed-boundary after Greene's residue optimization

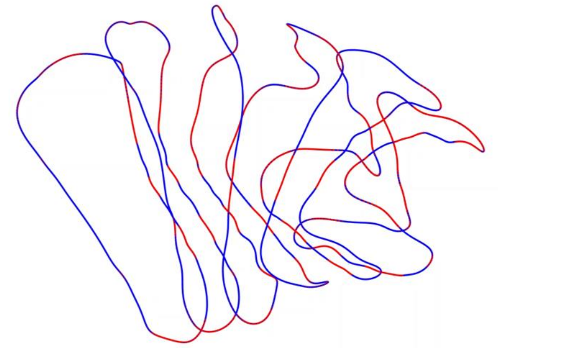
- FOCUS optimized single-filament coils reconstruct boundary well, but resonant field error induces sizeable 8/7 islands
- Subsequent coil optimizations are performed that target the 8/7 island's squared helical flux, f_w

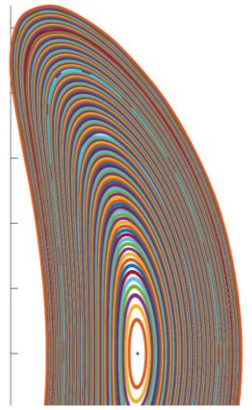




Free-boundary from original coils

- Coil shape gradient for helical flux used to minimize f_w





Free-boundary after helical flux optimization

- Original coils shown in red and helical flux optimized coils shown in blue. Small differences in coil geometry indicate high island sensitivity

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Future Work and Conclusions

Helical Flux Variations that do not Assume Periodic Field Lines Stay Fixed

- Adjoint formulation for helical flux variations is derived that does not assume the periodic field lines stay fixed

$$F = \int_0^{2\pi m_r}\mathbf{A}\cdot\dot{\mathbf{r}}\,d\varphi = \int_0^{2\pi m_r}\frac{1}{B^\varphi}\mathbf{A}\cdot\mathbf{B}\,d\varphi$$

- The Lagrangian, $\mathcal{L} = F + G + H$ is equal to F if $\mathbf{r}$ is a magnetic field line

$$\mathcal{L} = \int_0^{2\pi m_r}\frac{1}{B^\varphi}\mathbf{A}\cdot\mathbf{B}\,d\varphi + \int_0^{2\pi m_r}\lambda_R\left(\frac{dR}{d\varphi} - \frac{B^R}{B^\varphi}\right)d\varphi + \int_0^{2\pi m_r}\lambda_Z\left(\frac{dZ}{d\varphi} - \frac{B^Z}{B^\varphi}\right)d\varphi.$$

- By taking variations of $\mathcal{L}$ we find two coupled ODEs for the adjoint variables

$$\frac{d\lambda_R}{d\varphi} - \hat{\mathbf{R}}\cdot\nabla\left(\frac{1}{B^\varphi}\mathbf{A}\cdot\mathbf{B}\right) + \lambda_R\hat{\mathbf{R}}\cdot\nabla\left(\frac{B^R}{B^\varphi}\right) + \lambda_Z\hat{\mathbf{R}}\cdot\nabla\left(\frac{B^Z}{B^\varphi}\right) = 0$$

$$\frac{d\lambda_Z}{d\varphi} - \hat{\mathbf{Z}}\cdot\nabla\left(\frac{1}{B^\varphi}\mathbf{A}\cdot\mathbf{B}\right) + \lambda_R\hat{\mathbf{Z}}\cdot\nabla\left(\frac{B^R}{B^\varphi}\right) + \lambda_Z\hat{\mathbf{Z}}\cdot\nabla\left(\frac{B^Z}{B^\varphi}\right) = 0$$

- Using these adjoint variables, variations of $\mathcal{L}$ are independent of the magnetic field line location and if $\mathbf{r}$ is a magnetic field line then $\delta\mathcal{L} = \delta F$

$$\delta\mathcal{L} = \int_0^{2\pi m_r}\frac{\delta F}{\delta\mathbf{A}}\cdot\delta\mathbf{A}\,d\varphi + \int_0^{2\pi m_r}\left(\frac{\delta F}{\delta\mathbf{B}} + \frac{\delta G}{\delta\mathbf{B}} + \frac{\delta H}{\delta\mathbf{B}}\right)\cdot\delta\mathbf{B}\,d\varphi$$

Validating Variations are Linear

- The coil tolerance functional assumes variations are linear. If linear variations do not predict the actual variations accurately enough, higher order variations will need to be included in the coil tolerance functional
- To understand quadratic functional variations, consider the functional J that has an arbitrary integrand L

$$J = \int_a^b L(x, \mathbf{r}, \mathbf{r}')\,dx$$

- First and second order variations of the functional are given as

$$\delta J = \int_a^b\left(\frac{\partial L}{\partial\mathbf{r}_c} - \frac{d}{dx}\frac{\partial L}{\partial\mathbf{r}'}\right)\cdot\delta\mathbf{r}\,dx + \int_a^b\frac{1}{2}\delta\mathbf{r}\cdot\left(\delta\mathbf{r}\cdot\frac{\partial^2 L}{\partial\mathbf{r}^2}\right)dx + \frac{\partial L}{\partial\mathbf{r}'}\cdot\delta\mathbf{r}\Big|_a^b +$$
$$\int_a^b\left(\delta\mathbf{r}'\cdot\left(\delta\mathbf{r}\cdot\frac{\partial^2 L}{\partial\mathbf{r}\partial\mathbf{r}'}\right) + \frac{1}{2}\delta\mathbf{r}'\cdot\left(\delta\mathbf{r}'\cdot\frac{\partial^2 L}{\partial\mathbf{r}'^2}\right)\right)dx.$$

- To check if variations are linear, a subset of coil variations, with magnitudes below some tolerance, will be tested to see if the linear helical flux variations are sufficiently close to the actual helical flux variations

Coil Tolerance Maximization

- The coil tolerance can be maximized after an accurate coil tolerance is calculated
- Variations of coil tolerance may be difficult to calculate especially if the assumptions that variations are linear and periodic field lines stay fixed are poor

Conclusions

- The helical flux squared is an effective island width objective function
- The assumption that periodic magnetic field lines stay fixed under magnetic field variations is good enough to minimize the WISTELL-A 8/7 island
- A coil tolerance functional is derived, but includes two caveats
  - Periodic magnetic field lines stay fixed
  - Both functional variations are linear

[1] Imbert-Gerard, L. M. (2019). An introduction to symmetries in stellarators. *arXiv preprint arXiv:1908.05360*.
[2] Hudson, S. R. (2002). Eliminating islands in high-pressure free-boundary stellarator magnetohydrodynamic equilibrium solutions. *Physical review letters*, *89*(27), 275003.
[3] Bader, A. (2020). Advancing the physics basis for quasi-helically symmetric stellarators. *Journal of Plasma Physics*, *86*(5).