



## **Evolution of the plasma rotation and the radial electric field for a toroidal plasma in the Pfirsch–Schlüter and plateau regimes subject to a biased electrode**

M. Coronado and J. N. Talmadge

Citation: [Physics of Fluids B](#) **5**, 1200 (1993); doi: 10.1063/1.860910

View online: <http://dx.doi.org/10.1063/1.860910>

View Table of Contents: <http://scitation.aip.org/content/aip/journal/pofb/5/4?ver=pdfcov>

Published by the [AIP Publishing](#)

---

### **Articles you may be interested in**

[Pfirsch–Schlüter electric field in a tokamak](#)

*Phys. Plasmas* **16**, 044509 (2009); 10.1063/1.3124139

[Transport of a trace impurity in a dirty plasma in the Pfirsch–Schlüter regime](#)

*Phys. Fluids* **24**, 284 (1981); 10.1063/1.863357

[Transport of a multiple-ion species plasma in the Pfirsch–Schlüter regime](#)

*Phys. Fluids* **20**, 589 (1977); 10.1063/1.861917

[Impurity transport in tokamaks in the plateau to Pfirsch–Schlüter regimes](#)

*Phys. Fluids* **20**, 127 (1977); 10.1063/1.861701

[Impurity transport in the Pfirsch–Schlüter regime](#)

*Phys. Fluids* **17**, 1782 (1974); 10.1063/1.1694975

---

# Evolution of the plasma rotation and the radial electric field for a toroidal plasma in the Pfirsch-Schlüter and plateau regimes subject to a biased electrode

M. Coronado<sup>a)</sup> and J. N. Talmadge

*Department of Electrical and Computer Engineering, University of Wisconsin—Madison, Madison, Wisconsin 53706-1691*

(Received 19 August 1992; accepted 8 December 1992)

In this paper, the fluid equation approach is used to analyze the time evolution of the plasma rotation and the ambipolar electric field in a nonsymmetric toroidal plasma subject to an external biasing voltage induced by a probe. Under consideration is a plasma with low rotation speed in the Pfirsch-Schlüter or the plateau regime that includes the effects of a background neutral gas. A time-dependent charge conservation equation is used to determine the ambipolar electric field as a function of time. It is found that, after the application of the biasing voltage, the electric field and the plasma rotation change quickly and reach steady-state after a time inversely proportional to the sum of the momentum damping rates due to parallel viscosity and ion-neutral collisions. The steady state is characterized by a radial electric field and a plasma rotation that are proportional to the electric current flowing through the biasing probe. The direction of the plasma flow is determined by the relative magnitude of the momentum damping rates on the flux surface. From the steady-state solution, an expression for the radial electric conductivity is obtained, which includes the effect of collisions with neutrals as well as viscosity. Axisymmetric systems without neutrals are also discussed, which is a special case since there is no momentum damping in the toroidal direction. Here, the toroidal velocity increases continuously in time with the bias and never reaches steady state. Finally, a model for nonsymmetric magnetic fields is presented and the viscous damping rate, the radial conductivity and the spin-up rate for a plasma in the Pfirsch-Schlüter regime are calculated. As examples, the cases of the rippled tokamak and the classical and helically symmetric stellarators are evaluated.

## I. INTRODUCTION

The determination of the plasma rotation velocity and the ambipolar (radial) electric field in toroidal plasmas has been a controversial issue since the beginning of fusion research to the present. Based on neoclassical transport theory, the diffusion in axisymmetric systems is intrinsically ambipolar.<sup>1,2</sup> To determine the ambipolar electric field, it is necessary that calculations be done at higher order in the iterative scheme<sup>3</sup> or that external momentum losses or sources be included.<sup>4</sup> The buildup phase of the ambipolar field has been described by Hirshman.<sup>5</sup> It involves a dynamic nonambipolar phase, where the momentum of the plasma is damped in the poloidal direction at a fast rate by the viscosity and in the toroidal direction at a slow rate by weak processes such as charge exchange. In nonsymmetric systems, the diffusion is not intrinsically ambipolar<sup>6</sup> and the radial electric field can be calculated using the steady-state diffusion fluxes.<sup>7,8</sup> However, time-dependent equations should be used to analyze the solutions when the plasma is in a low collisionality regime since multivalued roots can appear.<sup>9</sup>

With the observation in tokamaks of the transition from the low to the high confinement regime (L-H tran-

sition) and the experimentally observed changes in the plasma rotation and the radial electric field, the problem of determining the ambipolar electric field and the plasma rotation in a tokamak has received renewed interest.<sup>10,11</sup> To artificially induce the L-H transition in tokamaks, strong radial electric fields have been set up through the application of an electrode bias.<sup>12,13</sup> In this paper, we do not aim for a description of the L-H transition, which requires the assumption of strong plasma rotation.<sup>10,14,15</sup> Instead, we develop a model using neoclassical fluid theory, assuming low rotation speed, to study the dynamics of the ambipolar electric field and the plasma rotation in an axisymmetric or nonsymmetric system after the application of a bias voltage with an electrode. A positive bias produces a radially inward electric current (i.e., outward electron flow) in the bias electrode. To maintain neutrality, a radially outward electric current should flow in the plasma in the region between the magnetic surface where the biasing electrode is located and the vacuum chamber or the limiter. After a fast transient, when the radial electric field and the plasma rotation increase, a steady state is reached because of momentum damping produced by viscosity and collisions with neutrals. In the new steady state, the radial electric field is related to the radial current and thus an expression for the radial electric conductivity can be derived.

<sup>a)</sup>Permanent address: Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, c.p. 70-543, 04510 México D.F.

By using an electrode to bias a toroidal plasma, we obtain a steady-state radial electric current. This radial current can just as well be produced by ions in loss orbits as in the model by Shaing,<sup>10</sup> nonambipolar electron losses as described by Itoh,<sup>16</sup> anomalous viscosity as assumed by Rozhanskii and Tendler,<sup>17</sup> or by the anomalous diffusion of suprathermal electrons in the model for auto-oscillating L–H transitions.<sup>18</sup>

In the moment approach to neoclassical transport theory, the fluid equations used to describe the time evolution of the ambipolar electric field in nonsymmetric systems have been briefly discussed by Shaing,<sup>19</sup> following the basic ideas for axisymmetric systems by Hirshman,<sup>5</sup> for the case of free relaxation to equilibrium without neutral gas effects, and without *external* radial electric currents. In this paper, we analyze the problem of the evolution in time and steady-state behavior of the plasma rotation and the ambipolar electric field in a nonsymmetric toroidal plasma in presence of an *external* radial current and a background neutral gas. In addition, we solve the equations for the case of low plasma rotation speed and a collisionality in the Pfirsch–Schlüter or plateau regimes where the drag due to the parallel viscosity increases linearly with the velocity. We use the fluid approach to neoclassical transport theory,<sup>20,21</sup> and neglect heat fluxes (i.e., temperature gradients), as well as electron viscosity, since the ratio of electron viscosity to ion viscosity is proportional to  $(m_e/m_i)^{1/2}$  for the regimes in consideration. We also assume that the density does not change during the time in which the electric field increases. This is not completely correct since both the ambipolar field and the density profile are based on the same process, which is particle diffusion. However, there is a fast time scale on which only the electric field changes and a slow scale at which both the density gradient and the electric field change together.<sup>9</sup> Our concern is mainly with the fast time scale.

Our paper is organized as follows; in Sec. II, we present the time-dependent moment approach and our scaling. In Sec. III, we deal with the problem of determining the radial current and plasma rotation in terms of the bias current, parallel viscosity, and collisions with neutrals. Section IV is devoted to the steady state, where we present expressions for the radial conductivity, ambipolar electric field, and plasma rotation. Section V treats the dynamics of the electric field and plasma rotation, and Sec. VI considers the application of the biasing voltage, where an expression for the relaxation time is derived. In Sec. VII, we evaluate the viscous damping, the electrical conductivity, and the relaxation time for a plasma in the Pfirsch–Schlüter regime using a model for an arbitrary magnetic field. This provides us with concise and useful formulas, which we then evaluate for the case of a tokamak with ripples, a helically symmetric stellarator, and a classical stellarator.

## II. THE MOMENT APPROACH AND THE MODEL

Consider a plasma composed of two charged species and neutral atoms in a toroidal magnetic field which may be axisymmetric or nonsymmetric. To describe the plasma

dynamics, we make use of the fluid approach to neoclassical transport theory.<sup>8,20,21</sup> The momentum balance equation is

$$m_a N_a \frac{\partial \mathbf{U}_a}{\partial t} + m_a N_a (\mathbf{U}_a \cdot \nabla) \mathbf{U}_a = e_a N_a \left( \mathbf{E} + \frac{\mathbf{U}_a \times \mathbf{B}}{c} \right) - \nabla p_a - \nabla \cdot \bar{\pi}_a + \mathbf{F}_a - m_a N_a \nu_{an} \mathbf{U}_a \quad (1)$$

and the continuity equation is given by

$$\frac{\partial N_a}{\partial t} + \nabla \cdot N_a \mathbf{U}_a = 0. \quad (2)$$

The subscript “*a*” accounts for the plasma species and  $m_a$ ,  $e_a$ ,  $N_a$ , and  $\mathbf{U}_a$  are the mass, electric charge, particle density, and fluid velocity, respectively,  $\mathbf{F}_a$  is the friction due to Coulomb collisions, and  $-m_a N_a \nu_{an} \mathbf{U}_a$  gives the drag due to the interaction of the charged particles with the neutrals, where  $\nu_{an}$  is the momentum damping rate. Also,  $p_a$  is the pressure, which we assume is given by  $p_a = N_a T_a$ , where  $T_a$  is the plasma temperature, and  $\bar{\pi}_a$  is the viscosity tensor that takes account of neoclassical effects. Here,  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic fields, respectively, which in principle have to be determined self-consistently using Maxwell’s equations and the moment equations.

To close the hierarchy of the moment equations in the two lowest-order moments, i.e., Eqs. (1) and (2), we neglect in this paper the heat flux, which requires a constant temperature  $\nabla T_a = 0$ . Also, we provide expressions for  $\mathbf{F}$  and  $\bar{\pi}_a$  in terms of the particle densities  $N_a$  and the macroscopic velocities  $\mathbf{U}_a$ . For the friction force, we use<sup>20</sup>

$$\mathbf{F}_i = m_i N_i \nu_{ie} (\mathbf{U}_e - \mathbf{U}_i), \quad (3)$$

where  $\nu_{ie}$  is the collision frequency of ions with electrons. Because of momentum conservation, it holds that  $\mathbf{F}_e = -\mathbf{F}_i$ , which means  $m_e N_e \nu_{ie} = m_i N_i \nu_{ei}$ . For the viscosity, we consider the expressions resulting from the parallel viscosity, which are given below in Sec. IV.

In this paper, we make use of the standard expansion scheme in the parameter gyroradius over characteristic-system-length,  $\epsilon \equiv R_g/L$ ,<sup>22</sup> (there are two parameters since the gyroradius is different for ions and electrons). We take the ordering:  $U_a/\nu_{th,a} \sim O(R_{g,a}/L)$  and  $V_E = cE/B \sim U_i \sim U_e$ , where  $\nu_{th,a}$  is the thermal speed of the particle species *a*, and  $V_E$  is the electric drift velocity. We consider a pressure anisotropy proportional to  $R_g/L$ , where  $L$  is a characteristic length of the system, thus  $|\nabla \cdot \bar{\pi}_a|/|\nabla p_a| \sim O(R_{g,a}/L)$ .<sup>22</sup> When large gradients exist in the plasma (a characteristic of the H regime), this length  $L$  may take values smaller than the torus minor radius. In this work, we assume that the pressure anisotropy remains small enough so that the ideal magnetohydrodynamic (MHD) equilibrium is still valid to lowest order. We also consider  $T_i \sim T_e$ ,  $\nu_{ie}/\omega_{gi} \lesssim O(R_{g,i}/L)$ ,  $\nu_{in}/\omega_{gi} \lesssim O(R_{g,i}/L)$ , and  $\nu_{en}/\omega_{ge} \lesssim O(R_{g,e}/L)$ , where  $\omega_{g,a}$  is the gyrofrequency of the plasma species *a*. For the time dependence, we assume that  $[(1/U_i)(\partial U_i/\partial t)]/\omega_{gi} \sim O(R_{g,i}/L)$  and  $\partial U_e/\partial t \sim \partial U_i/\partial t$ , together with

$(\partial N_a/\partial t)/(\nabla \cdot N_a U_a) \sim O(R_{ga}/L)$ . We also consider the presence of a loop voltage, whose electric field  $E_L$  is a first-order quantity.

With this ordering, the momentum balance equation to lowest order in  $(R_g/L)$  gives

$$e_a N_a^{(0)} \left( \mathbf{E}^{(0)} + \frac{1}{c} \mathbf{U}_a^{(0)} \times \mathbf{B}^{(0)} \right) = \nabla p_a^{(0)} \quad (4)$$

and the continuity equation yields

$$\nabla \cdot N_a^{(0)} \mathbf{U}_a^{(0)} = 0. \quad (5)$$

Here, the superscript (0) means lowest-order quantity.

By summing Eqs. (4) and (5) over the charged species and assuming quasineutrality, it follows

$$\frac{1}{c} \mathbf{J}^{(0)} \times \mathbf{B}^{(0)} = \nabla p^{(0)} \quad (6)$$

and

$$\nabla \cdot \mathbf{J}^{(0)} = 0, \quad (7)$$

where  $\mathbf{J}_a^{(0)} = \sum_a e_a N_a^{(0)} \mathbf{U}_a^{(0)}$  and  $p^{(0)} = \sum_a p_a^{(0)}$ . These equations together with the lowest-order Maxwell's equation,

$$\nabla \cdot \mathbf{E}^{(0)} = 4\pi \rho^{(0)} = 0, \quad (8)$$

$$\nabla \times \mathbf{E}^{(0)} = 0, \quad (9)$$

$$\nabla \cdot \mathbf{B}^{(0)} = 0, \quad (10)$$

$$\nabla \times \mathbf{B}^{(0)} = (4\pi/c) \mathbf{J}^{(0)}, \quad (11)$$

give the ideal magnetohydrodynamic equations that determine the plasma equilibrium. From Eq. (9) it follows that  $\mathbf{E}^{(0)}$  is electrostatic,  $\mathbf{E}^{(0)} = -\nabla \Phi^{(0)}$ . We assume there exists a solution of these equations with a magnetic field that has well-defined nested magnetic surfaces, which we label by the volume  $V$  that they enclose.

The first-order momentum balance equations contain resistivity, viscosity, and neutral drag. For the ions, it is given by

$$\begin{aligned} m_i N_i^{(0)} \frac{\partial \mathbf{U}_i^{(0)}}{\partial t} = & e N_i^{(1)} \left( \mathbf{E}^{(0)} + \frac{1}{c} \mathbf{U}_i^{(0)} \times \mathbf{B}^{(0)} \right) + e_i N_i^{(0)} \\ & \times \left( \mathbf{E}^{(1)} + \frac{1}{c} \mathbf{U}_i^{(1)} \times \mathbf{B}^{(0)} + \frac{1}{c} \mathbf{U}_i^{(0)} \times \mathbf{B}^{(1)} \right) \\ & - \nabla p_i^{(1)} - \nabla \cdot \bar{\pi}^{(0)} - m_i N_i^{(0)} \nu_{in} \mathbf{U}_i^{(0)} \\ & - m_i N_i^{(0)} \nu_{ie} (\mathbf{U}_i^{(0)} - \mathbf{U}_e^{(0)}), \end{aligned} \quad (12)$$

where the superscript (1) labels the first-order quantities. Since we are considering low rotation speeds, the inertia term is second order, and therefore we neglect it. For electrons, the momentum balance equation to first order [which contains terms of order  $(m_i/m_e)^{1/2} (R_{ge}/L)$ ] is

$$\begin{aligned} e N_e^{(1)} \left( \mathbf{E}^{(0)} + \frac{1}{c} \mathbf{U}_e^{(0)} \times \mathbf{B}^{(0)} \right) + e N_e^{(0)} \left( \mathbf{E}^{(1)} + \frac{1}{c} \mathbf{U}_e^{(1)} \right. \\ \left. \times \mathbf{B}^{(0)} + \frac{1}{c} \mathbf{U}_e^{(0)} \times \mathbf{B}^{(1)} \right) + \nabla p_e^{(1)} \\ + m_e N_e^{(0)} \nu_{ei} (\mathbf{U}_e^{(0)} - \mathbf{U}_i^{(0)}) = 0. \end{aligned} \quad (13)$$

The first-order electric field,  $\mathbf{E}^{(1)}$  in Eqs. (12) and (13), contains the toroidal electric field due to the loop voltage  $E_L$ .

The continuity equation takes the form

$$\frac{\partial N_a^{(0)}}{\partial t} + \nabla \cdot (N_a^{(0)} \mathbf{U}_a^{(1)} + N_a^{(1)} \mathbf{U}_a^{(0)}) = 0. \quad (14)$$

Equations (12)–(14) should be solved simultaneously with the first-order Maxwell's equations, which are

$$\nabla \times \mathbf{B}^{(1)} = \frac{4\pi}{c} \mathbf{J}^{(1)} + \frac{1}{c} \frac{\partial \mathbf{E}^{(0)}}{\partial t}, \quad (15)$$

$$\nabla \times \mathbf{E}^{(1)} = -\frac{1}{c} \frac{\partial \mathbf{B}^{(0)}}{\partial t}, \quad (16)$$

$\nabla \cdot \mathbf{E}^{(1)} = 4\pi \rho^{(1)} = 0$ , and  $\nabla \cdot \mathbf{B}^{(1)} = 0$ , where  $\rho^{(1)}$  is the first-order charge density that is equal to zero outside or inside the magnetic surface where the biasing probe is located. The first-order current density is  $\mathbf{J}^{(1)} = \sum_a e_a (N_a^{(0)} \mathbf{U}_a^{(1)} + N_a^{(1)} \mathbf{U}_a^{(0)})$ .

If we consider the *radial* component of Eq. (15) [i.e.,  $\nabla V \cdot$  Eq. (15)] and perform a volume integral in any region between two magnetic surfaces, divide it by the volume enclosed, and then take the limit when the thickness of this region vanishes, we obtain<sup>5</sup>

$$\frac{\partial \langle \mathbf{E}^{(0)} \cdot \nabla V \rangle}{\partial t} = -4\pi \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle, \quad (17a)$$

where  $\langle \nabla V \cdot \nabla \times \mathbf{B}^{(1)} \rangle = 0$  was used and  $\langle f \rangle$  represents the surface average of  $f$ . Equation (17a) describes a dynamic ambipolar condition. A positive radial current reduces the radial electric field, and this current changes the electric field in such a way to reduce the radial plasma current (assuming the current is proportional to the electric field). To maintain the plasma neutrality in steady state, the net radial current should vanish:

$$\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = 0. \quad (17b)$$

To simplify the mathematics, we will consider a plasma with a low  $\beta$  value, so that the feedback of the particle diffusion on the lowest-order magnetic field structure can be neglected. Therefore, in Eqs. (12)–(16), we set  $\mathbf{B}^{(1)} = 0$  and  $\partial \mathbf{B}^{(0)}/\partial t = 0$ . From here on we will drop the superscript (0) in  $\mathbf{B}^{(0)}$ .

### III. RADIAL CURRENT AND PLASMA SPIN-UP

Let us assume the magnetic field  $\mathbf{B}$  is given and therefore we can determine the ion and electron fluid velocity from Eqs. (4) and (5), as a function of the radial electric field,  $-\nabla \Phi^{(0)}$ , and particle density (recall that the temper-

ature gradient is assumed to be zero). The time behavior of these velocities is then determined by the time dependence in the electric field, since as discussed above, we are considering a fast relaxation phase, keeping the density constant in time.

By considering the component of Eq. (4) in the parallel direction (to  $\mathbf{B}$ ), assuming that quasineutrality holds and that magnetic surfaces are equipotentials, i.e.,  $\mathbf{B} \cdot \nabla \Phi^{(0)} = 0$ , we find that  $\mathbf{B} \cdot \nabla p_a^{(0)} = 0$ . Thus we can write  $\Phi^{(0)} = \Phi^{(0)}(V, t)$  and  $p_a = p_a(V)$ . We can write the perpendicular (to  $\mathbf{B}$ ) component of  $\mathbf{U}_a^{(0)}$  as

$$\mathbf{U}_{\perp, a}^{(0)} = -c \left( \Phi'^{(0)} + \frac{1}{e_a N_a^{(0)} p_a^{(0)}} \right) \frac{\nabla V \times \mathbf{B}}{B^2}, \quad (18)$$

where the prime indicates the derivative with respect to the volume  $V$ . Note that, to this order, the plasma flow is entirely in the magnetic surface,  $\mathbf{U}_a^{(0)} \cdot \nabla V = 0$ , and no diffusion is present.

The parallel component of  $\mathbf{U}_a^{(0)}$  can be determined from Eq. (5), i.e.,

$$\nabla \cdot \mathbf{U}_{\parallel, a}^{(0)} = -\nabla \cdot \mathbf{U}_{\perp, a}^{(0)} \quad (19)$$

together with Eq. (18) and  $p_a^{(0)} = N_a^{(0)} T_a$ .

To solve Eq. (19) for  $\mathbf{U}_{\parallel, a}^{(0)}$ , it is useful to employ Hamada coordinates<sup>23</sup>  $\{V, \theta, \xi\}$  in which the Jacobian is equal to one, and  $\theta$  and  $\xi$  are *poloidal* and *toroidal* angles varying from zero to one. In these coordinates, both the magnetic field

$$\mathbf{B} = B^\theta(V) \mathbf{e}_\theta + B^\xi(V) \mathbf{e}_\xi \quad (20)$$

and the plasma velocity of the two species

$$\mathbf{U}_a^{(0)} = U_a^\theta(V, t) \mathbf{e}_\theta + U_a^\xi(V, t) \mathbf{e}_\xi \quad (21)$$

are described by a straight line; i.e., their contravariant components are surface constants. A big advantage of this coordinate system over others is that the time dependence of the velocity appears only coupled with the coordinate  $V$  and not with  $\theta$  and  $\xi$ .

The solution of Eq. (19) in Hamada coordinates is<sup>24</sup>

$$\mathbf{U}_{\parallel, a}^{(0)} = c \left( \Phi'^{(0)} + \frac{1}{e_a N_a^{(0)} p_a^{(0)}} \right) \frac{B_\theta}{B^\xi B^2} \mathbf{B} + \lambda_a \mathbf{B}, \quad (22)$$

where  $B_\theta = \mathbf{B} \cdot \mathbf{e}_\theta$  is the poloidal covariant component of the magnetic field, which is a function of  $V$ ,  $\theta$ , and  $\xi$ . The first part of  $\mathbf{U}_{\parallel, a}^{(0)}$  makes  $\mathbf{U}_a^{(0)}$  divergence free, and therefore contains the so-called Pfirsch-Schlüter or return flows.<sup>24</sup> The second part is divergence free since  $\lambda_a$  is a (unknown) function of only  $V$  and time.

By combining Eqs. (18) and (22), we can write the contravariant components of  $\mathbf{U}_a^{(0)}$  as

$$U_a^\theta = c \left( \Phi'^{(0)} + \frac{1}{e_a N_a^{(0)} p_a^{(0)}} \right) \frac{1}{B^\xi} + \lambda_a B^\theta, \quad (23a)$$

$$U_a^\xi = \lambda_a B^\xi. \quad (23b)$$

As usual in the moment approach to neoclassical transport theory,<sup>8,20,21</sup> the constant  $\lambda_a$  is to be determined through the first-order parallel momentum balance equa-

tions, as we show below. This constant  $\lambda_a$  couples the poloidal and toroidal components of the velocity together.

Let us consider the scalar products of Eqs. (12) and (13) with  $\mathbf{B}$  and take the surface average. It results in

$$\begin{aligned} m_i N_i^{(0)} \frac{\partial}{\partial t} \langle \mathbf{B} \cdot \mathbf{U}_i^{(0)} \rangle \\ = - \langle \mathbf{B} \cdot \nabla \cdot \bar{\pi}_i \rangle - m_i N_i^{(0)} v_{in} \langle \mathbf{B} \cdot \mathbf{U}_i^{(0)} \rangle \\ + \langle \mathbf{B} \cdot \mathbf{F}_i \rangle + e N_i^{(0)} \langle \mathbf{E}_L \cdot \mathbf{B} \rangle \end{aligned} \quad (24)$$

and

$$\langle \mathbf{B} \cdot \mathbf{F}_e \rangle - e N_e^{(0)} \langle \mathbf{E}_L \cdot \mathbf{B} \rangle = 0, \quad (25)$$

where  $\mathbf{E}^{(1)} = -\nabla \Phi^{(1)} + \mathbf{E}_L$ , and the expression in Eq. (3) together with the identity  $\langle \mathbf{B} \cdot \nabla f \rangle = 0$  (valid for any scalar function  $f$ ) were used. Equation (25) is Ohm's law since  $\mathbf{F}_e = (m_i v_{ie}/e) \mathbf{J}$ ; hence, the parallel current is Ohmic and no bootstrap current is present (within this ordering the electron viscosity is neglected).<sup>24</sup> By adding Eqs. (24) and (25) we obtain

$$\begin{aligned} m_i N_i^{(0)} \frac{\partial}{\partial t} \langle \mathbf{B} \cdot \mathbf{U}_i^{(0)} \rangle = - \langle \mathbf{B} \cdot \nabla \cdot \bar{\pi}_i^{(0)} \rangle \\ - m_i N_i^{(0)} v_{in} \langle \mathbf{B} \cdot \mathbf{U}_i^{(0)} \rangle. \end{aligned} \quad (26)$$

Here, the friction force and the electric field  $\mathbf{E}_L$  drop out. Therefore the equations for the ions are completely decoupled from the equations for the electrons.

We consider now the poloidal component of the magnetic field defined as  $\mathbf{B}_p \equiv B^\theta(V) \mathbf{e}_\theta$  and take the scalar product of Eqs. (12) and (13) with  $\mathbf{B}_p$ . It should be pointed out here that  $\mathbf{B}_p$  is defined in Hamada coordinates and that it satisfies  $\nabla \cdot \mathbf{B}_p = 0$ . The surface average of  $\mathbf{B}_p \cdot [\text{Eq. (12)}]$  yields

$$\begin{aligned} m_i N_i^{(0)} \frac{\partial \langle \mathbf{B}_p \cdot \mathbf{U}_i^{(0)} \rangle}{\partial t} \\ = - B^\theta B^\xi \frac{e_i}{c} \langle \Gamma_i^{(1)} \cdot \nabla V \rangle - \langle \mathbf{B}_p \cdot \nabla \cdot \bar{\pi}_i^{(0)} \rangle \\ - m_i N_i^{(0)} v_{in} \langle \mathbf{B}_p \cdot \mathbf{U}_i^{(0)} \rangle + \langle \mathbf{B}_p \cdot \mathbf{F}_i^{(0)} \rangle; \end{aligned} \quad (27)$$

and the equivalent equation for the electrons gives

$$0 = B^\theta B^\xi \frac{e}{c} \langle \Gamma_e^{(1)} \cdot \nabla V \rangle + \langle \mathbf{B}_p \cdot \mathbf{F}_e \rangle, \quad (28)$$

where  $\Gamma_a^{(1)} \equiv N_a^{(0)} \mathbf{U}_a^{(1)} + N_a^{(1)} \mathbf{U}_a^{(0)}$  is the particle flow. In Eqs. (27) and (28),  $\langle \mathbf{B}_p \cdot \mathbf{E}_L \rangle = 0$  was used, since no net voltage difference appears with a turn in the poloidal direction.

We can now add these last two equations to find

$$m_i N_i^{(0)} \frac{\partial \langle \mathbf{B}_p \cdot \mathbf{U}_i^{(0)} \rangle}{\partial t} = -\frac{B^\theta B^\zeta}{c} \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle - \langle \mathbf{B}_p \cdot \nabla \cdot \bar{\pi}_i \rangle - m_i N_i^{(0)} v_{in} \langle \mathbf{B}_p \cdot \mathbf{U}_i^{(0)} \rangle. \quad (29)$$

Here,  $\mathbf{J}^{(1)} \cdot \nabla V$  is the radial electric current flowing in the plasma. In steady state, this current should vanish. This is not true, however, when an external mechanism driving a radial current is present, as, for example, in the case of a biasing probe, or for the radial currents involved in some of the L-H transition models<sup>10,16-18</sup> or for stochastic fluctuations that produce nonintrinsically ambipolar radial diffusion or for any other nonintrinsically ambipolar mechanism (such as other higher-order effects mentioned in Ref. 25). In these situations, the total net radial current should vanish to guarantee plasma neutrality, that is

$$\langle \mathbf{J}_{\text{tot}}^{(1)} \cdot \nabla V \rangle = \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle + \langle \mathbf{J}_{\text{ext}}^{(1)} \cdot \nabla V \rangle = 0. \quad (30)$$

This condition forces the neoclassical (internal) plasma currents  $\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle$  to adjust to a new value.

In a perfectly axisymmetric tokamak without neutrals, the neoclassical diffusion is intrinsically ambipolar, as mentioned above. This can also be seen straightforwardly from an equation equivalent to Eq. (29) with the toroidal component instead of the poloidal one. In steady state and without neutrals, it reads

$$\frac{B^\theta B^\zeta}{c} \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = \langle \mathbf{B}_T \cdot \nabla \cdot \bar{\pi}_i \rangle. \quad (31)$$

In axisymmetric plasmas,  $\langle \mathbf{B}_T \cdot \nabla \cdot \bar{\pi}_i \rangle = 0$ , and therefore  $\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = 0$ .<sup>26</sup> Thus, as mentioned in Ref. 25, a neoclassical tokamak in steady state cannot support any external radial current unless a nonintrinsically ambipolar diffusion mechanism is also available [like momentum exchange with neutrals in Eq. (29)].

When an external radial current is present, the time-dependent equation (17a) should be modified to have the total current instead of  $\mathbf{J}^{(1)}$ , i.e.,

$$\frac{\partial \langle \mathbf{E}^{(0)} \cdot \nabla V \rangle}{\partial t} = -4\pi (\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle + \langle \mathbf{J}_{\text{ext}}^{(1)} \cdot \nabla V \rangle), \quad (32)$$

and consequently, in steady state  $\langle \mathbf{J}_{\text{tot}} \cdot \nabla V \rangle = 0$ .

Let us now go back to Eq. (29) and discuss the plasma spin-up process due to the presence of external radial currents. Let us, for the moment, assume that viscosity is not present and that we are in a steady-state phase without a radial current, which implies  $\langle \mathbf{B}_p \cdot \mathbf{U}_i \rangle = 0$ . At a certain time, we switch on an external radial current. According to Eq. (30), the external current will require an internal current and therefore Eq. (29) describes a forced poloidal spin-up with damping due to collisions with neutrals. The new steady state is characterized by a nonvanishing poloidal rotation whose magnitude depends on  $\langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle$ . In reality, the problem is more complicated since instead of the steady-state condition (31), we should use Eq. (32). Thus the electric field appears explicitly in Eq. (29) and

therefore this equation may be seen as an equation for  $\Phi^{(0)}$  after substitution of  $\mathbf{U}_i$  in terms of  $\Phi^{(0)}$  using expressions in (23). This introduces another problem because these expressions involve the constant  $\lambda_p$ , which in turn is to be determined using Eq. (26). Thus Eqs. (26), (29), and (32) must be solved simultaneously. The same equation structure appears when we also take the viscosity into account, since in general, viscosity couples the poloidal and the parallel (or the toroidal) velocities via the magnetic field inhomogeneities.

#### IV. RADIAL CONDUCTIVITY AND STEADY-STATE AMBIPOLAR ELECTRIC FIELD AND PLASMA ROTATION

Let us now consider the steady state and find the relationship between the radial electric field and the radial current. In steady state, Eqs. (26) and (29) are

$$0 = \langle \mathbf{B} \cdot \nabla \cdot \bar{\pi}_i \rangle + m_i N_i^{(0)} v_{in} \langle \mathbf{B} \cdot \mathbf{U}_i \rangle, \quad (33)$$

and

$$-\frac{B^\theta B^\zeta}{c} \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = \langle \mathbf{B}_p \cdot \nabla \cdot \bar{\pi}_i \rangle + m_i N_i^{(0)} v_{in} \langle \mathbf{B}_p \cdot \mathbf{U}_i \rangle. \quad (34)$$

The first equation determines the surface constant  $\lambda_i$  from Eqs. (23); this means that the ion velocity is completely determined in terms of the electric field, the ion pressure, and the momentum losses. By substituting Eqs. (23) into Eq. (34), we obtain the radial current in terms of the electric field and the pressure, and from here, we obtain an expression for the radial electric conductivity.

For the viscosity, we only take the parallel viscosity, which is the dominant term in magnetized plasmas, and we neglect the perpendicular viscosity and the gyroviscosity.<sup>27</sup> The parallel viscosity describes the exchange of momentum between the plasma and the magnetic field occurring within magnetic surfaces (magnetic pumping) and does not contain the exchange of momentum between plasma regions in neighboring magnetic surfaces. Parallel viscosity involves poloidal and toroidal derivatives of the velocity (not radial derivatives) that can be evaluated in terms of poloidal and toroidal derivatives of the magnetic field. In the case of low rotation speeds, it is found that the parallel viscosity expressions are linear in the velocity.<sup>20,21,28</sup> In axisymmetric systems, the contribution of the parallel viscosity to the momentum balance in the direction of symmetry (toroidal direction) vanishes. If one ignores ion-neutral collisions, then there exists an instability in the toroidal rotation as described in the Appendix. Within the framework of neoclassical theory, it means that higher-order momentum dissipative terms, such as gyroviscosity and perpendicular viscosity, should be taken into account in the toroidal momentum balance.

The expressions we use for the parallel viscosity are

$$\langle \mathbf{B} \cdot \nabla \cdot \bar{\pi}_i \rangle = \mu_\theta U^\theta + \mu_\zeta U^\zeta, \quad (35a)$$

$$\langle \mathbf{B}_p \cdot \nabla \cdot \bar{\pi}_i \rangle = \mu_\theta^{(p)} U^\theta + \mu_\zeta^{(p)} U^\zeta, \quad (35b)$$

and

$$\langle \mathbf{B}_T \cdot \nabla \cdot \bar{\pi}_i \rangle = \mu_\theta^{(T)} U^\theta + \mu_\xi^{(T)} U^\xi. \quad (35c)$$

Note that the magnetic field structure is folded into the  $\mu$  coefficients. As will be shown explicitly in Sec. VII, expressions (35) for the Pfirsch-Schlüter regime satisfies  $B^\xi \mu_\xi^{(P)} = B^\theta \mu_\theta^{(T)}$  together with  $\mu_\theta = \mu_\theta^{(P)} + \mu_\theta^{(T)}$  and  $\mu_\xi = \mu_\xi^{(P)} + \mu_\xi^{(T)}$  (these relations also hold in the plateau regime). The advantage of using Hamada coordinates is apparent here, since the velocity components are surface constants and are completely decoupled from the geometry.

In describing Eqs. (35), the static assumption was made,<sup>5</sup> which means that ions should remain in quasithermal equilibrium. Strictly speaking, this restricts our model to relaxation times larger than  $\nu_{ii}^{-1}$ .

By using Eqs. (23) and (35a) in Eq. (33), it follows

$$\lambda_i = -c \left( \Phi' + \frac{1}{eN_i} p'_i \right) \frac{(\mu_\theta + m_i N_i \nu_{in} \langle B_\theta \rangle)}{B^\xi (B \cdot \mu + m_i N_i \nu_{in} \langle B^2 \rangle)}, \quad (36)$$

where  $B \cdot \mu = B^\theta \mu_\theta + B^\xi \mu_\xi$ , and the superscript (0) has been dropped from the lowest-order quantities.

Equations (23) for the ion velocity then become

$$U_i^\theta = c \left( \Phi' + \frac{1}{eN_i} p'_i \right) \frac{\mu_\xi + m_i N_i \nu_{in} \langle B_\xi \rangle}{(B \cdot \mu + m_i N_i \nu_{in} \langle B^2 \rangle)}, \quad (37a)$$

and

$$U_i^\xi = -c \left( \Phi' + \frac{1}{eN_i} p'_i \right) \frac{\mu_\theta + m_i N_i \nu_{in} \langle B_\theta \rangle}{(B \cdot \mu + m_i N_i \nu_{in} \langle B^2 \rangle)}. \quad (37b)$$

By substituting Eqs. (35b) and (21) in Eq. (34), it follows that

$$\begin{aligned} \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = & -\frac{cm_i N_i \langle B_p^2 \rangle}{B^\theta B^\xi B^\xi} \left[ (\nu_\theta^{(P)} + \nu_{in}) U^\theta \right. \\ & \left. + \left( \nu_\xi^{(P)} + i \frac{\langle \mathbf{B}_P \cdot \mathbf{B}_T \rangle}{\langle B_p^2 \rangle} \nu_{in} \right) U^\xi \right], \end{aligned} \quad (38)$$

where the definitions of  $\mathbf{B}_P$ ,  $\mathbf{B}_T$ , and  $t \equiv B^\theta/B^\xi$  (the rotational transform) were used, and the momentum damping rates due to poloidal viscosity are defined as

$$\nu_\theta^{(P)} \equiv \frac{\mu_\theta^{(P)} B^\theta}{m_i N_i \langle B_p^2 \rangle}, \quad \nu_\xi^{(P)} \equiv \frac{\mu_\xi^{(P)} B^\theta}{m_i N_i \langle B_p^2 \rangle}. \quad (39a)$$

It is appropriate to present here some momentum damping rates that we are going to use later. They are the damping rates due to the parallel viscosity

$$\nu_\theta \equiv \frac{\mu_\theta B^\xi}{m_i N_i \langle B^2 \rangle}, \quad \nu_\xi \equiv \frac{\mu_\xi B^\xi}{m_i N_i \langle B^2 \rangle}, \quad (39b)$$

and the damping rates due to toroidal viscosity

$$\nu_\theta^{(T)} \equiv \frac{\mu_\theta^{(T)} B^\xi}{m_i N_i \langle B_T^2 \rangle}, \quad \nu_\xi^{(T)} \equiv \frac{\mu_\xi^{(T)} B^\xi}{m_i N_i \langle B_T^2 \rangle}. \quad (39c)$$

For the Pfirsch-Schlüter and plateau regimes, it holds that  $\langle B_p^2 \rangle \nu_\xi^{(P)} = t^2 \langle B_T^2 \rangle \nu_\theta^{(T)}$  and  $\nu_\theta^{(P)}$  and  $\nu_\xi^{(T)}$  are always positive, but  $\nu_\theta^{(T)}$  may become negative in some cases.

By substituting Eqs. (37) into Eq. (38), using  $p' = \nabla p \cdot \mathbf{e}_v$  and  $\Phi' = -\mathbf{E} \cdot \mathbf{e}_v$  together with  $\mathbf{e}_v = \nabla V / |\nabla V|^2$ . We obtain

$$\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = \sigma_r \left( \mathbf{E} \cdot \nabla V - \frac{1}{eN_i} \nabla p_i \cdot \nabla V \right), \quad (40)$$

where  $\sigma_r$  is the plasma radial conductivity given by

$$\begin{aligned} \sigma_r = & \frac{c^2 m_i N_i}{(t\nu_\theta + \nu_\xi + \nu_{in}) |\nabla V|^2 (B^\theta B^\xi)^2} \left[ (\nu_\theta^{(P)} + \nu_{in}) \right. \\ & \times \left( \nu_\xi + \frac{\langle \mathbf{B} \cdot \mathbf{B}_T \rangle}{\langle B^2 \rangle} \nu_{in} \right) - \left( \nu_\xi^{(P)} + i \frac{\langle \mathbf{B}_P \cdot \mathbf{B}_T \rangle}{\langle B_p^2 \rangle} \nu_{in} \right) \\ & \left. \times \left( \nu_\theta + \frac{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{t \langle B^2 \rangle} \nu_{in} \right) \right]. \end{aligned} \quad (41)$$

The dimensions of  $\sigma_r$  are  $(\text{sec})^{-1}$  since we are using Gaussian units. Equation (40) gives the surface average of the radial current due to a radial electric field and a radial ion pressure gradient.

In the limiting case when collisions of ions with neutrals are negligible, we obtain

$$\sigma_r = \frac{c^2 m_i N_i \langle B_p^2 \rangle}{(t\nu_\theta + \nu_\xi) (B^\theta B^\xi)^2 |\nabla V|^2} (\nu_\theta^{(P)} \nu_\xi - \nu_\xi^{(P)} \nu_\theta). \quad (42)$$

By recalling the definition of the  $\nu$ 's, we can show that  $\sigma_r \geq 0$ . In axisymmetric systems, we have  $\langle \mathbf{B}_T \cdot \nabla \cdot \bar{\pi}_i \rangle = 0$  and  $\mu_\xi = \mu_\xi^{(P)} = \mu_\xi^{(T)} = \mu_\theta^{(T)} = 0$ , which implies  $\sigma_r = 0$ . This shows that, for axisymmetric systems that are in steady state, the neoclassical diffusion is intrinsically ambipolar and thus the radial current vanishes independently of the value of  $\Phi'$  and  $p'_i$ .

In the limiting case when viscosity effects are negligible compared to damping caused by collisions with neutrals, we obtain

$$\sigma_r = \frac{c^2 m_i N_i \nu_{in} (\langle B^2 \rangle \langle B_p^2 \rangle - \langle B^\theta B_\theta \rangle \langle \mathbf{B}_P \cdot \mathbf{B}_T \rangle)}{\langle B^2 \rangle |\nabla V|^2 (B^\theta B^\xi)^2}. \quad (43)$$

In the limiting case of a large-aspect-ratio tokamak, we can evaluate this expression using the corresponding Hamada coordinate<sup>29</sup> to obtain

$$\sigma_r = \left( \frac{e^2 N_i \nu_{in}}{m_i \omega_{gi}^2} \right) (1 + 2q^2), \quad (44)$$

where  $\omega_{gi} = eB/m_i c$  is the gyrofrequency, and  $q = 1/t$  is the safety factor. This result agrees with previous calculations.<sup>13,30</sup> The first factor gives the classical perpendicular conductivity, and the term  $(1 + 2q^2)$  is the Pfirsch-Schlüter enhancement factor due to toroidicity.

A steady-state plasma without external currents requires  $\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = 0$ , and therefore from Eq. (40) it follows

$$\Phi' = -(1/eN_i) p'_i, \quad (45)$$

which for constant temperature gives the Boltzmann relation  $N_i = N_0 \exp(-e\Phi/kT_i)$ . This condition in turn implies from Eq. (37) that  $U_i = 0$ , i.e., no average ion motion. As discussed before, Eq. (45) is invalid for an axisymmetric system without neutrals, since  $\sigma_r = 0$ ; thus a radial elec-

tric field in a *purely* neoclassical tokamak plasma without neutrals does not produce any radial current (diffusion is intrinsically ambipolar).

In steady state with an *external* radial current, Eqs. (30) and (40) imply that the electric field is given by

$$\Phi' = -\frac{1}{eN_i} p'_i + \frac{1}{\sigma_r} \left( \frac{\langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle}{|\nabla V|^2} \right). \quad (46)$$

In the limiting case of negligible viscosity and a large aspect-ratio tokamak with neutrals, we obtain

$$\frac{d\Phi}{dr} = -\frac{1}{eN_i} \frac{dp_i}{dr} + \frac{m_i \omega_{gi}^2}{e^2 N_i \nu_{in}} \frac{I_{\text{ext}}}{(1+2q^2)(4\pi^2 r R_0)^2}, \quad (47)$$

where  $\nabla V = 4\pi^2 r R_0 \hat{\phi}$  and  $\langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle / |\nabla V| = I_{\text{ext}} / 4\pi^2 r R_0$  were used; here,  $I_{\text{ext}}$  is the total *external* radial current, and  $r$  and  $R_0$  are the plasma radius and the tokamak major radius, respectively. The sign of the current carried should be properly taken into account when evaluating  $I_{\text{ext}}$ ; in a biasing experiment with positive voltage, there is a radial outward flow of electrons in the biased electrode, and therefore  $I_{\text{ext}}$  is negative. This gives a positive radial electric field, which accounts for an increment in the radial ion flow in the plasma that should compensate the electron flow in the biased electrode.

By substituting Eq. (46) in Eqs. (37) and the result in Eq. (21), we obtain that the ion velocity (plasma rotation) is proportional to the external current,

$$\mathbf{U}_i = \frac{\langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle}{eN_i |\nabla V|} \mathbf{K}_i, \quad (48)$$

where  $\mathbf{K}_i$  is a dimensionless vector whose magnitude and direction are determined by the ion momentum damping rates. It is given by

$$\mathbf{K}_i = \frac{eN_i c}{\sigma_r |\nabla V| B^\theta B^\zeta (i\nu_\theta + \nu_\zeta + \nu_{in})} \left[ \left( \nu_\zeta + \frac{\langle \mathbf{B}_T \cdot \mathbf{B} \rangle}{\langle \mathbf{B}^2 \rangle} \nu_{in} \right) \mathbf{B}_\rho - \left( i\nu_\theta + \frac{\langle \mathbf{B}_P \cdot \mathbf{B} \rangle}{\langle \mathbf{B}^2 \rangle} \nu_{in} \right) \mathbf{B}_T \right]. \quad (49)$$

In the limiting case of a large-aspect-ratio tokamak<sup>29</sup> with negligible viscosity, we obtain [to lowest order in  $(r/R_0)$ ]

$$\mathbf{K}_i \approx \frac{\omega_{gi}}{\nu_{in}(1+2q^2)} (\hat{\phi} - 2q \cos \vartheta \hat{\phi}), \quad (50)$$

where  $\hat{\phi}$  and  $\hat{\phi}$  are the poloidal and toroidal basis vectors in the standard orthogonal toroidal coordinate system (laboratory system).

## V. DYNAMICS OF THE AMBIPOLAR FIELD AND THE PLASMA ROTATION

Let us now consider the dynamics of the ambipolar electric field and the plasma rotation assuming we are in a steady state, and at a given time we start externally driving a radial current. As shown in Eq. (29), this radial current produces a change in poloidal velocity, which works against the damping caused by the neutrals and the viscosity. The viscosity in turn involves the toroidal (or the parallel) component of the velocity, and therefore couples Eq.

(29) to Eq. (26). These equations should be written in terms of the electric potential using Eq. (37) and solved together with the dynamic ambipolar condition, Eq. (32), where  $\langle \mathbf{E}^{(0)} \cdot \nabla V \rangle = -\Phi' |\nabla V|^2$ . Recall that we are assuming that the plasma density keeps constant during this process. Without this assumption, we would need to solve simultaneously the three equations mentioned above, plus the electron radial transport equation and the continuity equations for ions and electrons, which is beyond the scope of this paper.

By using  $\mathbf{B}_P \cdot \mathbf{U}_i = B^\theta (g_{\theta\theta} U^\theta + g_{\theta\zeta} U^\zeta)$ , with  $g_{\theta\theta} = \mathbf{e}_\theta \cdot \mathbf{e}_\theta$  and  $g_{\theta\zeta} = \mathbf{e}_\theta \cdot \mathbf{e}_\zeta$ , and employing Eq. (35b) together with expressions (23) and Eq. (32) we obtain the following equation for  $\Phi'$  and  $\lambda_i$  (which is the constant part of the parallel ion velocity):

$$a_1(V) \frac{\partial \Phi'}{\partial t} + a_2(V) \frac{\partial \lambda_i}{\partial t} + b_1(V) \Phi' + b_2(V) \lambda_i = C_1(V, t), \quad (51)$$

where  $\Phi = \Phi(V, t)$ ,  $\lambda_i = \lambda_i(V, t)$ , and

$$a_1 = \left( 1 + \frac{(B^\theta B^\zeta)^2 |\nabla V|^2}{4\pi c^2 m_i N_i \langle B_P^2 \rangle} \right), \quad a_2 = \frac{B^\theta B^\zeta \langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{c \langle B_P^2 \rangle}, \quad (52a)$$

$$b_1 = \nu_\theta^{(P)} + \nu_{in},$$

$$b_2 = \frac{B^\theta B^\zeta}{c} \left( \nu_\theta^{(P)} + q \nu_\zeta^{(P)} + \frac{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{\langle B_P^2 \rangle} \nu_{in} \right), \quad (52b)$$

$$C_1 = \frac{(B^\theta B^\zeta)^2}{c^2 m_i N_i \langle B_P^2 \rangle} \langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle - (\nu_\theta^{(P)} + \nu_{in}) \frac{p'_i}{eN_i}. \quad (52c)$$

By writing  $\mathbf{B} \cdot \mathbf{U}_i = B_\theta U^\theta + B_\zeta U^\zeta$  and using Eq. (35a) and expressions (23) in Eq. (26) we obtain an equation similar to Eq. (51). This is

$$a_3(V) \frac{\partial \Phi'}{\partial t} + a_4(V) \frac{\partial \lambda_i}{\partial t} + b_3(V) \Phi' + b_4(V) \lambda_i = C_2(V), \quad (53)$$

where

$$a_3 = \frac{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{\langle B^2 \rangle}, \quad a_4 = \frac{B^\theta B^\zeta}{c}, \quad (54a)$$

$$b_3 = i\nu_\theta + \frac{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{\langle B^2 \rangle} \nu_{in}, \quad b_4 = \frac{B^\theta B^\zeta}{c} (i\nu_\theta + \nu_\zeta + \nu_{in}), \quad (54b)$$

$$C_2 = - \left( i\nu_\theta + \frac{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{\langle B^2 \rangle} \nu_{in} \right) \frac{p'_i}{eN_i}. \quad (54c)$$

In this equation set, the constants  $a$ 's are geometrical factors, the constants  $b$ 's are damping rates, and the constants  $C$ 's play the role of sources. It describes a forced motion with damping. To solve these equations, we write them in matrix form:

$$\mathcal{A} \frac{d\mathbf{X}}{dt} + \mathcal{B} \mathbf{X} = \mathbf{C}. \quad (55)$$



For very general types of toroidal configurations, it holds that  $\mathcal{A}^{-1}$  exists, and we can write

$$\frac{d\mathbf{X}}{dt} = \mathcal{D}\mathbf{X} + \mathbf{S}, \quad (56)$$

where  $\mathcal{D} = -\mathcal{A}^{-1}\mathcal{B}$  and  $\mathbf{S} = \mathcal{A}^{-1}\mathbf{C}$ . The components of the vector  $\mathbf{S}$  are

$$S_1 = (a_4 C_1 - a_2 C_2) / \Delta, \quad (57a)$$

$$S_2 = (a_1 C_2 - a_3 C_1) / \Delta,$$

and the matrix entries of  $\mathcal{D}$  are

$$\begin{aligned} d_{11} &= (a_2 b_3 - a_4 b_1) / \Delta, \\ d_{12} &= (a_2 b_4 - a_4 b_2) / \Delta, \\ d_{21} &= (a_3 b_1 - a_1 b_3) / \Delta, \\ d_{22} &= (a_3 b_2 - a_1 b_4) / \Delta, \end{aligned} \quad (57b)$$

where

$$\begin{aligned} \Delta &\equiv \det(\mathcal{A}) \\ &= (B^\theta B^z / c) [1 - \langle \mathbf{B} \cdot \mathbf{B}_p \rangle^2 / \langle B^2 \rangle \langle B_p^2 \rangle] \\ &\quad + (B^\theta B^z |\nabla V|)^2 / 4\pi c^2 m_i N_i \langle B_p^2 \rangle]. \end{aligned}$$

The solution of the homogeneous part of Eq. (56) gives

$$\mathbf{X}_H(t) = P_1 \xi_1 + P_2 \xi_2, \quad (58)$$

where  $P_1$  and  $P_2$  are constants, and

$$\xi_j = e^{\gamma_j t} \begin{pmatrix} 1 \\ (\gamma_j - d_{11}) / d_{12} \end{pmatrix}, \quad (59)$$

with  $j=1$  or  $2$ . Here,  $\gamma_1$  and  $\gamma_2$  are growth (or relaxation) rates that are determined from the characteristic equation  $(d_{11} - \gamma)(d_{22} - \gamma) = d_{12} d_{21}$ ; it gives

$$\begin{aligned} \frac{\gamma_1}{\gamma_2} &= \frac{1}{2} (d_{11} + d_{22}) \\ &\pm \sqrt{\frac{1}{4} (d_{11} + d_{22})^2 - (d_{11} d_{22} - d_{12} d_{21})}. \end{aligned} \quad (60)$$

Although, in general, the possibility of having toroidal configurations that give complex roots cannot be ruled out, we find that, for the standard configurations, the roots are real with  $(d_{11} + d_{22}) < 0$ , leading to negative  $\gamma_1$  and  $\gamma_2$  with  $|\gamma_1| < |\gamma_2|$ . Also,  $\gamma_1$  and  $\gamma_2$  give the relaxation rates of the ambipolar electric field and the plasma flows;  $\gamma_1$  describes the slower relaxation time and  $\gamma_2$  is the faster one, which correspond to two different directions on the magnetic surface.

The general solution of Eq. (56) can be obtained from Eq. (58) by allowing  $P_1$  and  $P_2$  to depend on the time, and determining them by using Eq. (56). It results for  $\mathbf{X} = (X_1, X_2)$  that

$$X_1(t) = \frac{1}{\gamma_2 - \gamma_1} [M_1(t) + M_2(t)] + X_{1,0}(t), \quad (61a)$$

$$\begin{aligned} X_2(t) &= \frac{1}{(\gamma_2 - \gamma_1) d_{12}} [(\gamma_1 - d_{11}) M_1(t) \\ &\quad + (\gamma_2 - d_{11}) M_2(t)] + X_{2,0}(t), \end{aligned} \quad (61b)$$

where

$$M_1(t) = e^{\gamma_1 t} \int_{t_0}^t [-d_{12} S_2 + (\gamma_2 - d_{11}) S_1] e^{-\gamma_1 \tau} d\tau, \quad (62a)$$

$$M_2(t) = e^{\gamma_2 t} \int_{t_0}^t [d_{12} S_2 - (\gamma_1 - d_{11}) S_1] e^{-\gamma_2 \tau} d\tau, \quad (62b)$$

and  $X_{1,0}$  and  $X_{2,0}$  describe the transients associated with the initial conditions, i.e.,

$$X_{1,0}(t) = P_{1,0} e^{\gamma_1(t-t_0)} + P_{2,0} e^{\gamma_2(t-t_0)}, \quad (63a)$$

$$\begin{aligned} X_{2,0}(t) &= \frac{1}{d_{12}} [(\gamma_1 - d_{11}) P_{1,0} e^{\gamma_1(t-t_0)} \\ &\quad + (\gamma_2 - d_{11}) P_{2,0} e^{\gamma_2(t-t_0)}]. \end{aligned} \quad (63b)$$

The evolution in time of the electric field,  $\Phi' = X_1(t)$ , and the parallel ion velocity,  $\lambda_i = X_2(t)$ , is described by Eqs. (61). It involves the initial conditions,  $\Phi'(t_0)$  and  $\lambda_i(t_0)$ , the effective damping rates  $\gamma_1$  and  $\gamma_2$ , and the driving forces  $C_1$  and  $C_2$  [i.e.,  $S_1$  and  $S_2$  in Eqs. (62)]. The external radial current appears in  $C_1$ , and it can contain a time dependence. For times  $\Delta t \equiv t - t_0$  much larger than  $|\gamma_1|^{-1}$ , the effect of the initial conditions disappears. The time dependence of the external current determines the behavior of the radial electric field and the plasma rotation. When this current remains finite, the electric field and the rotation reach a steady state (except for axisymmetric systems without neutrals, where  $\gamma_1 = 0$ ), as will be shown in the next section. In the case where  $\gamma_1$  and  $\gamma_2$  are complex, the solutions are given by the real part of Eqs. (61); for  $\text{Re}(\gamma) < 0$  the steady state is reached after damped oscillations.

## VI. BIAS-PROBE EXPERIMENTS

Let us now consider a toroidal plasma with an electrode at a given magnetic surface  $V = V_0$ , as shown in Fig. 1. At a specific time  $t = t_1$  the electrode is biased to a given voltage with respect to the vacuum chamber. We want to describe the behavior in time of the ambipolar electric field and the plasma rotation after the time  $t = t_1$  by assuming that, prior to the biasing, the plasma is in steady-state (this means  $t_1 - t_0 \gg |\gamma_1|^{-1}$ ).

For simplicity, we model the external current  $I_{\text{ext}}$  as a step function located at  $t = t_1$  (see Fig. 2) and assume that  $\gamma_1$  and  $\gamma_2$  are real. From (52c), we obtain

$$C_1 = C_{1,0} + H(t - t_1) \delta C_1, \quad (64)$$

where  $C_{1,0} = C_1(I_{\text{ext}} = 0)$ , and  $\delta C_1 = (B^\theta B^z)^2 \langle J_{\text{ext}} \cdot \nabla V \rangle / c^2 m_i N_i \langle B_p^2 \rangle$  is constant in time, and  $H(t - t_1)$  is the step function.

The time behavior of  $\Phi'$  and  $\lambda_i$  can be obtained from Eqs. (61) with  $X_{1,0} = X_{2,0} = 0$  and

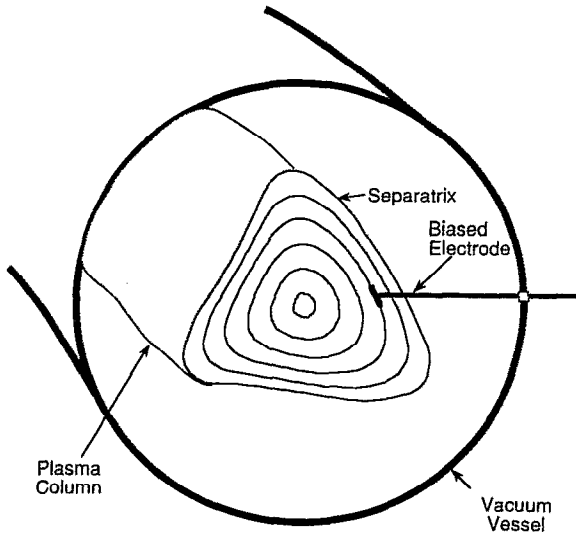


FIG. 1. A radial electric field is induced by a probe located inside the plasma.

$$M_1 = \gamma_1^{-1} [d_{12}S_{2,0} - (\gamma_2 - d_{11})S_{1,0}] + \delta C_1 (\gamma_1 \Delta)^{-1} [d_{12}a_3 + (\gamma_2 - d_{11})a_4] \times (1 - e^{\gamma_1(t-t_1)})H(t-t_1), \quad (65a)$$

$$M_2 = \gamma_2^{-1} [-d_{12}S_{2,0} + (\gamma_1 - d_{11})S_{1,0}] - \delta C_1 (\gamma_2 \Delta)^{-1} [d_{12}a_3 + (\gamma_1 - d_{11})a_4] \times (1 - e^{\gamma_2(t-t_1)})H(t-t_1), \quad (65b)$$

where  $S_{1,0} \equiv S_1(I_{\text{ext}}=0)$ ,  $S_{2,0} \equiv S_2(I_{\text{ext}}=0)$ , and  $e^{\gamma_j(t-t_0)} = 0$  were used. For  $t < t_1$ , we obtain the steady-state solution

$$\Phi' = (1/\gamma_1\gamma_2\Delta)(b_4C_{1,0} - b_2C_2), \quad (66a)$$

$$\lambda_i = (1/\gamma_1\gamma_2\Delta)(-b_3C_{1,0} + b_1C_2), \quad (66b)$$

and the change in time with respect to the values at  $t \lesssim t_1$  is given by

$$\Delta\Phi'(t) = \frac{-\delta C_1}{(\gamma_2 - \gamma_1)\gamma_1\gamma_2\Delta} \{ \gamma_2 [d_{12}a_3 + (\gamma_2 - d_{11})a_4] \times (1 - e^{\gamma_1 t}) - \gamma_1 [d_{12}a_3 + (\gamma_1 - d_{11})a_4] \times (1 - e^{\gamma_2 t}) \} \quad (67a)$$

and

$$\Delta\lambda_i(t) = \frac{-\delta C_1}{(\gamma_2 - \gamma_1)\gamma_1\gamma_2\Delta} \{ \gamma_2 [(\gamma_1 - d_{11})a_3 - d_{21}a_4] \times (1 - e^{\gamma_1 t}) - \gamma_1 [(\gamma_2 - d_{11})a_3 - d_{21}a_4] \times (1 - e^{\gamma_2 t}) \}. \quad (67b)$$

A schematic plot of  $\Phi'$  in time is shown in Fig. 2. After a rapid increase at  $t = t_1$ , it reaches a constant value in a time

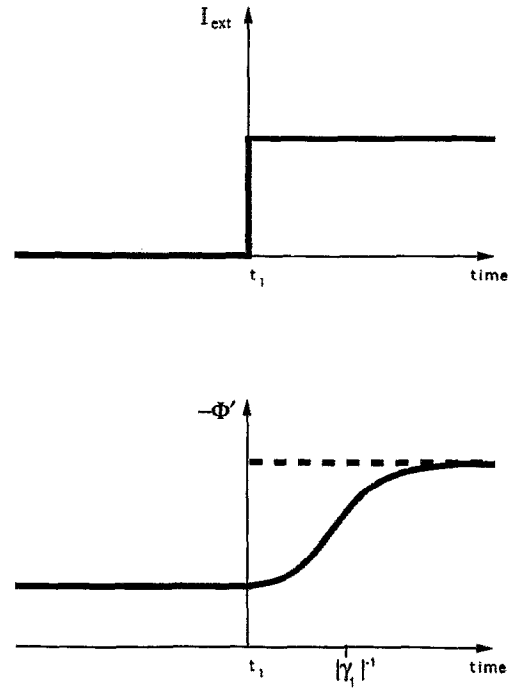


FIG. 2. The dynamics of the radial electric field after the electrode is biased at  $t = t_1$ . For  $t < t_1$ , the system is in steady state with zero biasing current, and for  $t \gg |\gamma_1|^{-1}$  the system reaches a steady state with finite biasing current.

on the order of  $|\gamma_1|^{-1}$ . From Eq. (67b), we see that  $\lambda_i$  has a similar time dependence. The net change in  $\Phi'$  and  $\lambda_i$  is given by

$$\Delta\Phi'(t \rightarrow \infty) = (b_4/\gamma_1\gamma_2\Delta)\delta C_1, \quad (68a)$$

$$\Delta\lambda_i(t \rightarrow \infty) = -(b_3/\gamma_1\gamma_2\Delta)\delta C_1, \quad (68b)$$

which together with Eqs. (66) gives the final value of  $\Phi'$  and  $\lambda_i$ . These values lead to the steady-state solution evaluated in Sec. IV and given by Eq. (46). (Notice that  $\gamma_1\gamma_2\Delta = b_1b_4 - b_3b_2$ ). The evolution of the ion poloidal and toroidal velocities can be obtained by substituting Eqs. (67) in Eqs. (23).

The general expression for the relaxation rates  $\gamma_1$  and  $\gamma_2$  in terms of the damping rates due to viscosity and collisions with neutrals is obtained from Eq. (60). If we define

$$v_1 \equiv v_\theta^{(P)} + (1 + I_0)(tv_\theta + v_\zeta) - \delta_1(v_\theta^{(P)} + qv_\zeta^{(P)}) - i\delta_2v_\theta, \quad (69)$$

we can write the full expression for  $\gamma_1$  and  $\gamma_2$  as

$$\left. \begin{matrix} \gamma_1 \\ \gamma_2 \end{matrix} \right\} = - \left\{ v_{in} + \frac{v_1 - I_0 v_{in}}{2\hat{\Delta}} \mp \left[ \left( \frac{v_1 - I_0 v_{in}}{2\hat{\Delta}} \right)^2 + \frac{v_{in} I_0 (tv_\theta + v_\zeta) + v_\zeta^{(P)} v_\theta - v_\theta^{(P)} v_\zeta}{\hat{\Delta}} \right]^{1/2} \right\}, \quad (70)$$

where  $\hat{\Delta} \equiv (c/B^\theta B^\zeta)\Delta = 1 + I_0 - \delta_1\delta_2$ ,  $\delta_1 \equiv \langle \mathbf{B} \cdot \mathbf{B}_P \rangle / \langle B^2 \rangle$ ,  $\delta_2 \equiv \langle \mathbf{B} \cdot \mathbf{B}_P \rangle / \langle B_P^2 \rangle$ , and  $I_0 = (B^\theta B^\zeta/c)^2 |\nabla V|^2 / 4\pi m_e N_i \langle B_P^2 \rangle$ . Here,  $\delta_1 < 1$ ,  $\delta_2 \geq 1$  and  $\delta_1\delta_2 < 1$ . Also,  $I_0$  is a dimensionless parameter that appears because of the time

derivative of the electric field in the charge conservation equation, Eq. (32), and explicitly emerges in the coefficient  $a_1$  in Eq. (52a). The value of  $I_0$  is, in general, very small, i.e., of the order of  $10^{-4}$  or even smaller. Thus we can neglect it and approximate  $\gamma_1$  and  $\gamma_2$  by

$$\left. \begin{aligned} \gamma_1 \\ \gamma_2 \end{aligned} \right\} = -v_{in} - \frac{v_1}{2\hat{\Delta}} \pm \left[ \left( \frac{v_1}{2\hat{\Delta}} \right)^2 + \frac{v_\xi^{(P)} v_\theta - v_\theta^{(P)} v_\xi}{\hat{\Delta}} \right]^{1/2}, \quad (71)$$

where  $\hat{\Delta}$  is a quantity slightly less than 1. The time it takes the plasma to completely reach steady state is determined by  $|\gamma_1|^{-1}$  (or its real part in the case  $\gamma_1$  is complex, which gives a damped oscillatory behavior before reaching steady state). This relaxation rate  $\gamma_1$  can be written as

$$\gamma_1 = -v_{in} - v_{visc}, \quad (72)$$

where

$$v_{visc} = v_1/2\hat{\Delta} - [(v_1/2\hat{\Delta})^2 + (v_\xi^{(P)} v_\theta - v_\theta^{(P)} v_\xi)/\hat{\Delta}]^{1/2}.$$

In the limiting case that viscosity is negligible with respect to collisions with neutrals,  $\gamma_1 = -v_{in}$ . In the case of axisymmetric systems, it holds that  $v_\xi = v_\xi^{(P)} = 0$ , and therefore  $\gamma_1 = -v_{in}$ . The case of axisymmetric high-temperature plasmas without neutrals yields  $\gamma_1 = 0$ , which when substituted in Eq. (62) gives a linear increase of the plasma rotation and the radial electric field as a function of time; i.e., no steady-state solution is possible. This result is physically correct since no damping is present in the axisymmetric direction and the poloidal and toroidal velocities are coupled. This coupling might appear doubtful at first glance at the poloidal momentum balance equation, Eq. (29), since only poloidal components of  $U_i$  seem to be present, however, the term  $\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle$  provides a coupling with the time derivative of the electric field [see Eq. (32)], which, in turn, couples with the toroidal velocity in Eq. (26) via Eqs. (23), as shown in the Appendix.

## VII. PFIRSCH-SCHLÜTER VISCOSITY DAMPING RATES IN THE LARGE-ASPECT-RATIO LIMIT

The radial conductivity, and the time behavior of the plasma rotation and the ambipolar electric field calculated in the above sections are expressed in terms of the viscosity damping rates in Eq. (39). In this section, we evaluate these damping rates for the case when the plasma is in the Pfirsch-Schlüter collisional regime and the magnitude of the magnetic field has the form

$$B = B_0 \left( 1 - \epsilon_T \cos \vartheta - \sum_{n,m} \epsilon_{m,n} \cos(m\vartheta + n\phi) \right), \quad (73)$$

where  $\epsilon_T \ll 1$ ,  $\epsilon_{m,n} \ll 1$ , and  $\vartheta$  and  $\phi$  are laboratory poloidal and toroidal angles varying between 0 and  $2\pi$ .

In the Pfirsch-Schlüter regime, the viscosity in Eqs. (35) and (39) is given by<sup>21</sup>

$$\langle \mathbf{A} \cdot \nabla \cdot \bar{\pi}_i \rangle = \frac{\mu_0 p_i}{v_{ii}} \left\langle \left( \frac{\mathbf{A} \cdot \nabla B}{B} \right) \left( \frac{\mathbf{U}_i \cdot \nabla B}{B} \right) \right\rangle, \quad (74)$$

where  $\mu_0 = 4.095$ ,  $\mathbf{A}$  may be  $\mathbf{B}$ ,  $\mathbf{B}_p$ , or  $\mathbf{B}_T$  (or any other vector satisfying  $\nabla \cdot \mathbf{A} = 0$  and  $\mathbf{A} \cdot \nabla V = 0$ ), and  $v_{ii}$  is the ion-ion collision frequency. To evaluate expression (74),

we make use of the Hamada coordinates derived in the large-aspect-ratio limit.<sup>29</sup> Within this model, the magnetic surfaces are given by concentric circles with radius  $r$ . The surface average of a quantity  $F(\vartheta, \phi)$  is given by

$$\langle F \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\phi F(\vartheta, \phi) \left( 1 + \frac{r}{R_0} \cos \vartheta \right), \quad (75)$$

where  $R = R_0 + r \cos \vartheta$  and  $R_0$  is the major radius. To evaluate  $\mathbf{A} \cdot \nabla B$  or  $\mathbf{U} \cdot \nabla B$ , we make use of the fact that  $\mathbf{A}$  and  $\mathbf{U}$  are vectors on the magnetic surface, i.e.,  $\mathbf{A} \cdot \hat{r} = \mathbf{U} \cdot \hat{r} = 0$ , and allow  $\mathbf{B}$  to vary within the surface. Thus

$$\mathbf{U} \cdot \nabla B = \left( \frac{\mathbf{U} \cdot \hat{\vartheta}}{r} \right) \frac{\partial B}{\partial \vartheta} + \left( \frac{\mathbf{U} \cdot \hat{\phi}}{R} \right) \frac{\partial B}{\partial \phi}, \quad (76)$$

where, instead of the vector  $\mathbf{U}$ , we may also use the vector  $\mathbf{A}$ . In order to write Eq. (74) in the form of Eq. (35), we should write  $\mathbf{U} \cdot \hat{\vartheta}$  and  $\mathbf{U} \cdot \hat{\phi}$  in terms of the Hamada contravariant components  $U^\theta$  and  $U^\xi$ . By using the results of Ref. 29, we can write

$$U^\theta = \mathbf{U} \cdot \nabla \theta = (1 + \epsilon_T \cos \vartheta) \mathbf{U} \cdot \left( \frac{\hat{\vartheta}}{2\pi r} \right) \quad (77a)$$

and

$$U^\xi = \mathbf{U} \cdot \nabla \xi \approx 2q\epsilon_0 \left( \frac{\mathbf{U} \cdot \hat{\vartheta}}{2\pi r} \right) \cos \vartheta + \mathbf{U} \cdot \left( \frac{\hat{\phi}}{2\pi R} \right), \quad (77b)$$

with  $\epsilon_0 = r/R_0$ . Thus, to lowest order in  $\epsilon_0$  and  $\epsilon_T$ , it holds

$$\mathbf{U} \cdot \nabla B = 2\pi \left( \frac{\partial B}{\partial \vartheta} U^\theta + \frac{\partial B}{\partial \phi} U^\xi \right), \quad (78)$$

and we can therefore approximate

$$\langle \mathbf{A} \cdot \nabla \cdot \bar{\pi}_i \rangle = \frac{4\pi^2 \mu_0 p_i}{v_{ii}} \langle (k_1 A^\theta + k_2 A^\xi) (k_1 U^\theta + k_2 U^\xi) \rangle, \quad (79)$$

where

$$k_1 = \epsilon_T \sin \vartheta + \sum_{m,n} \epsilon_{m,n} m \sin(m\vartheta + n\phi),$$

$$k_2 = \sum_{m,n} \epsilon_{m,n} n \sin(m\vartheta + n\phi),$$

and  $A^\theta$ ,  $A^\xi$ ,  $U^\theta$ , and  $U^\xi$  are surface constants. After evaluating the surface average, we obtain that

$$\langle \mathbf{A} \cdot \nabla \cdot \bar{\pi}_i \rangle = \frac{4\pi^2 \mu_0 p_i}{v_{ii}} [(\alpha_P A^\theta + \alpha_C A^\xi) U^\theta + (\alpha_C A^\theta + \alpha_T A^\xi) U^\xi], \quad (80)$$

where  $\alpha_P$  and  $\alpha_T$  give a measure of the magnetic field inhomogeneity in the poloidal and toroidal directions, respectively, and are given by

$$\alpha_P \equiv \langle k_1^2 \rangle = \frac{1}{2} \left( \epsilon_T^2 + \sum_{m,n} m^2 \epsilon_{m,n}^2 \right), \quad (81a)$$

$$\alpha_T \equiv \langle k_2^2 \rangle = \frac{1}{2} \sum_{m,n} n^2 \epsilon_{m,n}^2 \quad (81b)$$

together with

$$\alpha_C \equiv \langle k_1 k_2 \rangle = \frac{1}{2} \sum_{m,n} n m \epsilon_{m,n}^2. \quad (81c)$$

Expression (80) has the form appearing in Eq. (35) with  $\mathbf{A} = \mathbf{B}$ ,  $\mathbf{B}_P$ , or  $\mathbf{B}_T$ . From this, we can obtain the  $\mu$  coefficients and, consequently, the viscous damping rates in Eq. (39). To evaluate them we again use the Hamada coordinates from Ref. 29. In these coordinates, we have to lowest order

$$\langle B_P^2 \rangle = \epsilon_0^2 B_0^2 (1 + 2q^2)/q^2, \quad \langle \mathbf{B}_P \cdot \mathbf{B} \rangle = \epsilon_0^2 B_0^2/q^2, \quad (82a)$$

$$\langle B_T^2 \rangle = \langle B^2 \rangle = \langle \mathbf{B} \cdot \mathbf{B}_T \rangle = B_0^2, \quad \langle \mathbf{B}_P \cdot \mathbf{B}_T \rangle = -2\epsilon_0^2 B_0^2, \quad (82b)$$

$$B^\theta = B_0/2\pi q R_0, \quad \langle B_\theta \rangle = 2\pi r \epsilon_0 B_0/q, \quad (82c)$$

$$B^\xi = B_0/2\pi R_0, \quad B_\xi = 2\pi R_0 B_0, \quad (82d)$$

$$|\nabla V| = 4\pi^2 r R_0, \quad (82e)$$

and therefore

$$\nu_\theta^{(P)} = \frac{\nu_0 \alpha_P}{(1 + 2q^2) \epsilon_0^2}, \quad \nu_\xi^{(P)} = \frac{\nu_0 \alpha_C}{(1 + 2q^2) \epsilon_0^2}, \quad (83)$$

$$\nu_\theta^{(T)} = \nu_0 \alpha_C, \quad \nu_\xi^{(T)} = \nu_0 \alpha_T, \quad (84)$$

and

$$\nu_\theta = \nu_0 (\alpha_P/q + \alpha_C), \quad \nu_\xi = \nu_0 (\alpha_C/q + \alpha_T), \quad (85)$$

where  $\nu_0$  is a basic viscous frequency given by

$$\nu_0 = \frac{4.095 p_i}{\nu_{ii} m_i N_i R_0^2}. \quad (86)$$

We can now give approximate expressions for the radial conductivity  $\sigma_r$  in Eq. (41) and for the damping rate  $\gamma_1$  in Eq. (72). By using Eq. (82) and keeping only lowest-order terms, we can write

$$\sigma_r = \frac{c^2 m_i N_i (1 + 2q^2)}{B_0^2} \left( \frac{(\nu_\theta^{(P)} + \nu_{in})(\nu_\xi + \nu_{in}) - (\nu_\xi^{(P)} - [2q/(1 + 2q^2)] \nu_{in})(\nu_\theta + t \epsilon_0^2 \nu_{in})}{(i \nu_\theta + \nu_\xi + \nu_{in})} \right) \quad (87)$$

and

$$\nu_{\text{visc}} = \frac{1}{2} \{ \nu_1 - [\nu_1^2 + 4(\nu_\xi^{(P)} \nu_\theta - \nu_\theta^{(P)} \nu_\xi)]^{1/2} \} \quad (88)$$

with

$$\nu_1 = \nu_\theta^{(P)} + [2q/(1 + 2q^2)] \nu_\theta + \nu_\xi. \quad (89)$$

The first factor in  $\sigma_r$  is the same as in expression (44).

### A. The tokamak with magnetic ripple

We can evaluate  $\sigma_r$  and the relaxation rates  $\nu_{\text{visc}}$  for the case of a tokamak with magnetic ripple, where

$$B = B_0 [1 - \epsilon_T \cos \vartheta - \epsilon_R \cos(n\varphi)]. \quad (90)$$

In this case,  $\alpha_P = \frac{1}{2} \epsilon_T^2$ ,  $\alpha_T = \frac{1}{2} n^2 \epsilon_R^2$ , and  $\alpha_C = 0$ , thus, by taking  $\epsilon_T \approx \epsilon_0$ , it follows

$$\nu_\theta^{(P)} = \frac{1}{2} \nu_0 / (1 + 2q^2), \quad (91a)$$

$$\nu_\theta^{(T)} = \nu_\xi^{(P)} = 0, \quad (91b)$$

$$\nu_\xi^{(T)} = \frac{1}{2} \nu_0 n^2 \epsilon_R^2. \quad (91c)$$

Without collisions with neutrals, the radial conductivity is given by

$$\sigma_r = \frac{c^2 m_i N_i \nu_0}{2 B_0^2} \left( \frac{n^2 \epsilon_R^2}{\epsilon_T^2/q^2 + n^2 \epsilon_R^2} \right), \quad (92)$$

and the relaxation rates are given by

$$\gamma_1 = -\frac{1}{2} n^2 \epsilon_R^2 \nu_0, \quad (93a)$$

$$\gamma_2 = -\frac{1}{2} \nu_0 / (1 + 2q^2). \quad (93b)$$

The first rate gives the momentum relaxation in the toroidal direction and the second one is the relaxation rate in the poloidal direction.

### B. The helically symmetric stellarator

Let us now evaluate the case of a helically symmetric stellarator, where

$$B = B_0 [1 - \epsilon_H \cos(m\vartheta + n\varphi)], \quad (94)$$

and include the effect of neutral atoms. In this case,  $\alpha_P = \frac{1}{2} m^2 \epsilon_H^2$ ,  $\alpha_T = \frac{1}{2} n^2 \epsilon_H^2$ , and  $\alpha_C = \frac{1}{2} n m \epsilon_H^2$ , hence

$$\nu_\theta^{(P)} = \frac{m^2 \nu_0 \epsilon_H^2}{2(1 + 2q^2) \epsilon_0^2}, \quad (95a)$$

$$\nu_\theta = \frac{1}{2} \nu_0 m^2 \epsilon_H^2 \left( t + \frac{n}{m} \right), \quad (95b)$$

$$\nu_\xi^{(P)} = \frac{n}{m} \nu_\theta^{(P)}, \quad \text{and} \quad \nu_\xi = \frac{n}{m} \nu_\theta. \quad (95c)$$

Thus, if no neutral collisions are present, we obtain results similar to the tokamak case, where  $\sigma_r = 0$  and one of the relaxation rates is zero, which is a consequence of the symmetry of the magnetic field. In the case where neutrals are present,  $m \neq 0$  and  $n \lesssim m$ , we can approximate the radial conductivity by

$$\sigma_r = \frac{c^2 m_i N_i (1+2q^2) v_{in}}{B_0^2} \times \left( \frac{1 + (v_0/v_{in}) [m^2/2(1+2q^2)] (\epsilon_H^2/\epsilon_0^2)}{1 + \frac{1}{2} (v_0/v_{in}) (n+tm)^2 \epsilon_H^2} \right) \quad (96)$$

and the relaxation rates by

$$\gamma_1 = -v_{in} - \frac{v_0 m^2 \epsilon_H^2}{2(1+2q^2) \epsilon_0^2}, \quad (97a)$$

$$\gamma_2 = -v_{in}. \quad (97b)$$

### C. The classical stellarator

We finally evaluate the case of a classical stellarator, where

$$B = B_0 [1 - \epsilon_T \cos \vartheta - \epsilon_S \cos(m\vartheta + n\varphi)] \quad (98)$$

without the effect of collisions with neutrals. Here,  $\alpha_P = \frac{1}{2}(\epsilon_T^2 + m^2 \epsilon_S^2)$ ,  $\alpha_T = \frac{1}{2}n^2 \epsilon_S^2$ , and  $\alpha_C = \frac{1}{2}nm \epsilon_S^2$ , which gives

$$v_\theta^{(P)} = \frac{v_0}{2(1+2q^2) \epsilon_0^2} (\epsilon_T^2 + m^2 \epsilon_S^2), \quad (99a)$$

$$v_\theta = \frac{1}{2} v_0 \left[ t \epsilon_T^2 + m^2 \epsilon_S^2 \left( t + \frac{n}{m} \right) \right], \quad (99b)$$

$$v_\xi^{(P)} = \frac{nm v_0}{2(1+2q^2) \epsilon_0^2} \epsilon_S^2, \quad (99c)$$

$$v_\xi = \frac{nm v_0}{2} \epsilon_S^2 \left( t + \frac{n}{m} \right). \quad (99d)$$

The radial conductivity is given by

$$\sigma_r = \frac{c^2 m_i N_i v_0 n^2 \epsilon_T^2 \epsilon_S^2}{2 \epsilon_0^2 B_0^2 [t \epsilon_T^2 + m^2 \epsilon_S^2 (t + (n/m)^2)]} \quad (100)$$

and for  $m, n \ll \epsilon_S^{-1}$  and  $m \neq 0$  the relaxation rates are

$$\gamma_1 = -\frac{v_0}{2(1+2q^2) \epsilon_0^2} (\epsilon_T^2 + m^2 \epsilon_S^2), \quad (101a)$$

$$\gamma_2 = -\frac{v_0}{2 \epsilon_0^2} n^2 \epsilon_T^2 \epsilon_S^2. \quad (101b)$$

It is interesting to compare the results of a classical stellarator and the helically symmetric stellarator. Without neutrals, Eq. (96) gives  $\sigma_r = 0$  and (97b) gives  $\gamma_2 = 0$ , which are results obtained from Eqs. (100) and (101b), respectively, by making  $\epsilon_T = 0$ . The same results are obtained by making  $\epsilon_S = 0$  since we recover axisymmetry.

### VIII. SUMMARY AND CONCLUSIONS

In this paper, we have analyzed the dynamics of the ambipolar electric field and the plasma rotation in a non-symmetric toroidal plasma in the plateau and Pfirsch-Schlüter regimes when a bias voltage is applied through an electrode inside the plasma. With the turn-on of the bias voltage, a radial current is made to flow. The momentum in the surface is unbalanced making the plasma rotate and

the ambipolar electric field change, until a steady state is reached because of the presence of momentum damping caused by parallel viscosity and collisions with neutrals. There are two momentum relaxation rates associated with any magnetic surface, which define the direction of strongest and weakest momentum damping. A perfect axisymmetric high-temperature tokamak is, to lowest order, intrinsically ambipolar and one of these rates goes to zero, i.e., a steady state does not exist. In this case, the inclusion of mechanisms of toroidal momentum damping (such as perpendicular or gyroviscosity) are essential to reach a steady state. We have derived an expression for the radial conductivity that involves a standard part due to ion-neutral collisions and a new part due to parallel viscosity. We also evaluate the steady-state ambipolar electric field and the plasma rotation, which are directly proportional to the magnitude of the current flowing through the biasing probe. We finally derive very useful and simple expressions for the viscous damping rates for a large-aspect-ratio non-symmetric system and the relaxation time, the radial conductivity, and the plasma rotation have been evaluated.

### ACKNOWLEDGMENTS

This work was supported by U.S. Department of Energy Grants No. DE-FG02-86ER53216 and No. DE-FG02-85ER53198 and by the Dirección General de Asuntos del Personal Académico, National University of Mexico.

### APPENDIX: AXISYMMETRIC SYSTEMS

In this appendix, we discuss the equations that describe the dynamics of the poloidal and toroidal velocity in an axisymmetric system. In these systems,  $\mathbf{e}_\xi$  is parallel to  $\nabla \zeta$ , i.e.,  $\mathbf{e}_\xi = R^2 \nabla \zeta$ , and therefore  $g_{\xi\theta} = \mathbf{e}_\xi \cdot \mathbf{e}_\theta = 0$ , with  $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$  the metric tensor.

In this case,

$$\mathbf{B} \cdot \mathbf{U} = B^\theta U_\theta + B^\xi U_\xi = B^\theta g_{\theta\theta} U^\theta + B^\xi g_{\xi\xi} U^\xi \quad (A1)$$

and

$$\mathbf{B}_P \cdot \mathbf{U} = B^\theta g_{\theta\theta} U^\theta. \quad (A2)$$

Additionally, in an axisymmetric system  $\langle \mathbf{B}_T \cdot \nabla \cdot \bar{\pi} \rangle = 0$ , and therefore

$$\langle \mathbf{B} \cdot \nabla \cdot \bar{\pi} \rangle = \langle \mathbf{B}_P \cdot \nabla \cdot \bar{\pi} \rangle = \mu_\theta^{(P)} U^\theta. \quad (A3)$$

With these simplifications, the poloidal momentum balance equation given by Eq. (29) yields

$$m_i N_i B^\theta \langle g_{\theta\theta} \rangle \frac{\partial U^\theta}{\partial t} = -\frac{B^\theta B^\xi}{c} \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle - (\mu_\theta^{(P)} + m_i N_i v_{in} B^\theta \langle g_{\theta\theta} \rangle) U^\theta. \quad (A4)$$

In a similar way, we can write the parallel momentum balance equation, Eq. (26). If we subtract this last equation from Eq. (A4), we obtain an equation for the toroidal velocity

$$m_i N_i B^{\zeta} \langle g_{\zeta\zeta} \rangle \frac{\partial U^{\zeta}}{\partial t} = \frac{B^{\theta} B^{\zeta}}{c} \langle \mathbf{J}^{(1)} \cdot \nabla V \rangle - m_i N_i v_{in} B^{\zeta} \langle g_{\zeta\zeta} \rangle U^{\zeta}. \quad (\text{A5})$$

Equations (A4) and (A5) are coupled through the charge conservation equation, Eq. (32),

$$\langle \mathbf{J}^{(1)} \cdot \nabla V \rangle = - \langle \mathbf{J}_{\text{ext}}^{(1)} \cdot \nabla V \rangle + \frac{|\nabla V|^2}{4\pi} \frac{\partial \Phi'}{\partial t} \quad (\text{A6})$$

via the time derivative of the electric field. By using Eq. (23) and assuming the pressure is constant in time, we can obtain

$$\frac{\partial \Phi'}{\partial t} = \frac{B^{\zeta}}{c} \left( \frac{\partial U^{\theta}}{\partial t} - t \frac{\partial U^{\zeta}}{\partial t} \right). \quad (\text{A7})$$

If we substitute Eq. (A6) and Eq. (A7) in Eqs. (A4) and (A5), we obtain

$$(1 + I_0) \frac{\partial U^{\theta}}{\partial t} - i I_0 \frac{\partial U^{\zeta}}{\partial t} = \frac{4\pi c I_0}{B^{\zeta} |\nabla V|^2} \langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle - (\nu_{\theta}^{(P)} + \nu_{in}) U^{\theta} \quad (\text{A8})$$

and

$$(1 + k I_0) \frac{\partial U^{\zeta}}{\partial t} - \frac{I_0}{t} k \frac{\partial U^{\theta}}{\partial t} = \frac{-4\pi c I_0}{B^{\theta} |\nabla V|^2} k \langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle - \nu_{in} U^{\zeta}, \quad (\text{A9})$$

where  $k \equiv \langle B_p^2 \rangle / \langle B_T^2 \rangle$ , with  $B_p^2 = B^{\theta} B^{\theta} g_{\theta\theta}$ ,  $B_T^2 = B^{\zeta} B^{\zeta} g_{\zeta\zeta}$ , and  $I_0$  is the quantity defined near Eq. (70). These last two equations show that  $U^{\theta}$  and  $U^{\zeta}$  are coupled, and that an external radial current (bias current) affects both. However, assuming  $|\partial U^{\theta} / \partial t| \sim |\partial U^{\zeta} / \partial t|$  and since  $I_0 \ll 1$ , we can write

$$\frac{\partial U^{\zeta}}{\partial t} = \frac{-4\pi c I_0}{B^{\theta} |\nabla V|^2} k \langle \mathbf{J}_{\text{ext}} \cdot \nabla V \rangle - \nu_{in} U^{\zeta}. \quad (\text{A10})$$

The first term on the right-hand side of Eq. (A10) might be very small. However, when no ion-neutral collisions are present, i.e.,  $\nu_{in} = 0$ , it determines the behavior of the toroidal velocity, which would grow indefinitely if a constant external radial current was present. The poloidal velocity, however, has a net damping due to viscosity in the poloidal direction (magnetic pumping)  $\nu_{\theta}^{(P)}$ , which remains finite when the ion-neutral collisions vanish, leading to a constant value of the poloidal velocity for times much larger than  $(\nu_{\theta}^{(P)})^{-1}$ .

From the above analysis, it can be concluded that, in an axisymmetric system, the leading term of the viscosity tensor, i.e., the parallel viscosity, gives no contribution to the toroidal momentum damping. Therefore, in a perfect tokamak without neutrals, the toroidal velocity would in-

crease continuously. Real tokamaks do reach steady state because the transport may also involve other nonintrinsic ambipolar mechanisms such as collisions with neutrals or viscous damping due to the perpendicular viscosity or the gyroviscosity. Comparison of the experimental toroidal momentum damping rates in tokamaks with theoretical models that include gyroviscosity have indicated good agreement.<sup>31-33</sup> In our treatment, we have not included the perpendicular or gyroviscosity since they are negligible in comparison with the parallel viscosity for the case of non-symmetric systems.

<sup>1</sup>L. M. Kovrizhnykh, Sov. Phys. JETP **29**, 475 (1969).

<sup>2</sup>P. H. Rutherford, Phys. Fluids **13**, 482 (1970).

<sup>3</sup>M. N. Rosenbluth, P. H. Rutherford, J. B. Taylor, E. A. Frieman, and L. M. Kovrizhnykh, *Plasma Physics and Controlled Fusion Research* (International Atomic Energy Agency, Vienna, 1971), Vol. 1, p. 445; R. D. Hazeltine, Phys. Fluids **17**, 961 (1974).

<sup>4</sup>W. M. Stacey, Jr. and D. J. Sigmar, Phys. Fluids **22**, 2000 (1979).

<sup>5</sup>S. P. Hirshman, Nucl. Fusion **18**, 917 (1978).

<sup>6</sup>E. A. Frieman, Phys. Fluids **13**, 490 (1970).

<sup>7</sup>H. E. Mynick and W. N. G. Hitchon, Nucl. Fusion **23**, 1053 (1983).

<sup>8</sup>M. Coronado and H. Wobig, Phys. Fluids **30**, 3171 (1987).

<sup>9</sup>D. E. Hastings, Phys. Fluids **29**, 536 (1986).

<sup>10</sup>K. C. Shaing, E. C. Crume, Jr., and W. A. Houlberg, Phys. Fluids B **2**, 1492 (1990).

<sup>11</sup>R. J. Groebner, K. H. Burrell, and R. P. Seraydarian, Phys. Rev. Lett. **64**, 3015 (1990).

<sup>12</sup>R. J. Taylor, M. L. Brown, B. D. Fried, H. Grote, J. R. Liberate, G. J. Morales, and P. Pribyl, Phys. Rev. Lett. **63**, 2365 (1989).

<sup>13</sup>R. R. Weynants and R. J. Taylor, Nucl. Fusion **30**, 945 (1990).

<sup>14</sup>F. L. Hinton and S. K. Wong, Phys. Fluids **28**, 3082 (1985).

<sup>15</sup>W. M. Stacey, Jr., A. W. Bailey, D. J. Sigmar, and K. C. Shaing, Nucl. Fusion **25**, 463 (1985).

<sup>16</sup>S. I. Itoh and K. Itoh, Nucl. Fusion **29**, 1031 (1989).

<sup>17</sup>V. A. Rozhanskii and M. B. Tendler, JETP Lett. **53**, 82 (1991); L. G. Askinazi, V. E. Golant, S. V. Lebedev, V. A. Rozhanskij, and M. Tendler, Nucl. Fusion **32**, 271 (1992).

<sup>18</sup>I. A. Voitsekhovich, D. A. Kislov, V. V. Parail, and K. A. Razumova, "Autooscillating L-H transitions under ECCD in T-10," to appear in Sov. J. Plasma Phys.

<sup>19</sup>K. C. Shaing, Phys. Fluids **29**, 2231 (1986).

<sup>20</sup>S. P. Hirshman and D. J. Sigmar, Nucl. Fusion **21**, 1079 (1981).

<sup>21</sup>K. C. Shaing and J. D. Callen, Phys. Fluids **26**, 3315 (1983).

<sup>22</sup>F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **42**, 239 (1976).

<sup>23</sup>W. D. D'haeseleer, W. N. G. Hitchon, J. D. Callen, and J. L. Shohet, *Flux Coordinates and Magnetic Field Structure* (Springer-Verlag, Berlin, 1991).

<sup>24</sup>M. Coronado and H. Wobig, Phys. Fluids B **4**, 1294 (1992).

<sup>25</sup>Y. B. Kim and F. L. Hinton, Phys. Fluids B **4**, 278 (1992).

<sup>26</sup>In an axisymmetric system,  $\mathbf{B}_T$  is given by  $B^{\zeta} R^2 \nabla_{\zeta}$  and not by  $B^{\zeta} \nabla_{\zeta}$ , as incorrectly taken in Ref. 25. Therefore  $\nabla \cdot \mathbf{B}_T = \nabla \cdot \mathbf{B}_p = 0$ , and  $\langle \mathbf{B}_T \cdot \nabla \cdot \vec{\pi} \rangle = 0$ .

<sup>27</sup>W. M. Stacey, Jr. and D. J. Sigmar, Phys. Fluids **28**, 2800 (1985).

<sup>28</sup>M. Coronado and H. Wobig, Phys. Fluids **29**, 527 (1986).

<sup>29</sup>M. Coronado and J. Galindo Trejo, Phys. Fluids B **2**, 530 (1990).

<sup>30</sup>M. Okabayashi and S. Yoshikawa, Phys. Rev. Lett. **29**, 1725 (1972).

<sup>31</sup>K. H. Burrell, R. J. Groebner, H. St. John, and R. P. Seraydarian, Nucl. Fusion **28**, 3 (1988).

<sup>32</sup>W. M. Stacey, Nucl. Fusion **31**, 31 (1991).

<sup>33</sup>D. R. Jackson and W. M. Stacey, Bull. Am. Phys. Soc. **37**, 1448 (1992).