Stability analysis using the SIESTA equilibrium code

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Overview

○SIESTA (Scalable Iterative Equilibrium Solver for Toroidal Applications) is a 3D nonlinear MHD equilibrium solver capable of resolving islands in confinement devices in an accurate and scalable manner.

○The presence of a numerical nullspace of the Hessian matrix has important convergence implications for SIESTA. The structure of the nullspace eigenmodes has been calculated and compare favorably with expectations. The calculations were done on a three field-period stellarator.

○A stability analysis has been carried out for a CDX-U tokamak that is Mercier unstable (shown here) and a Solov'ev configuration. Problems were detected with the LAPACK eigensolver dgeev for use on our poorly-conditioned Hessian matrix.

○The eigenspectrum work is important not only for improving the convergence of SIESTA, but also for future work on the physical effect of islands on the MHD Alfven spectrum. The impact of islands on the presence of gap modes and the existence of Alfven modes in 3D equilibria will be studied in future work.

Basic equations of SIESTA

Ideal MHD energy (target function for minimization):

$$W = \int \frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} dVol$$

The force in the unperturbed state which is not necessarily yet in equilibrium is given by: $\vec{F}_0 = \vec{J}_0 \times \vec{B}_0 - \nabla p_0$

For any plasma displacement, the linearized system becomes the following:

Perturbed, linear system $\delta\vec{F} = \delta\vec{J} \times \vec{B}_0 + \vec{J}_0 \times \delta\vec{B} - \nabla\delta p$

Faraday's law $\delta\vec{B} = \nabla \times \left(\vec{\xi} \times \vec{B}_0\right)$

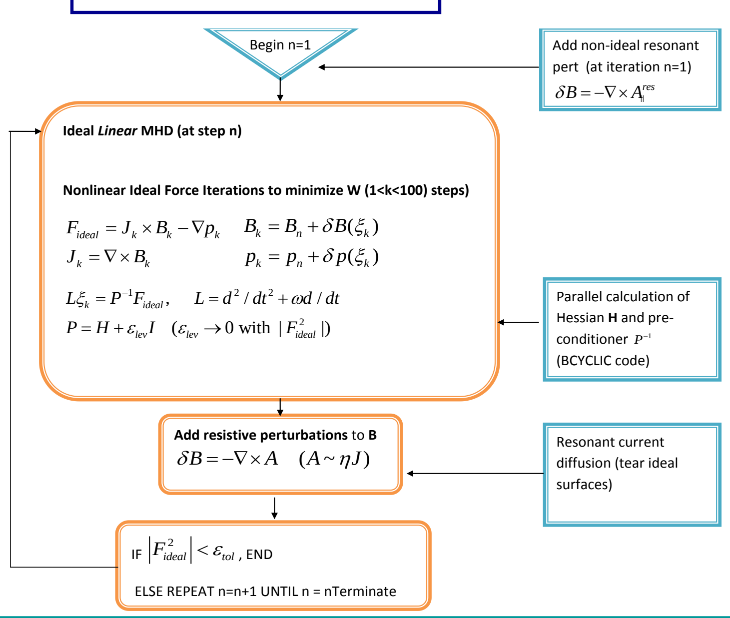
Ampere's law $\delta\vec{J} = \frac{1}{\mu_0}\left(\nabla \times \delta\vec{B}\right)$

Mass conservation $\delta p = (\gamma - 1)\vec{\xi} \cdot \nabla p_0 - \gamma\nabla \cdot \left(p_0\vec{\xi}\right)$

To find equilibrium, minimize nonlinear force $\vec{F} \equiv \vec{J} \times \vec{B} - \nabla p = \vec{0}$ and update fields.

Linear equation to be solved iteratively by GMRES: $\vec{H}\vec{\xi} = -\vec{F}_{res}$, $\frac{\partial\vec{F}}{\partial\vec{\xi}}\vec{\xi} = -\vec{F}_{res}$

At a VMEC or SIESTA equilibrium, the eigenvalues of the SIESTA Hessian correspond to actual physical stable/unstable modes. VMEC equilibria will in general contain unstable modes in the SIESTA context, because VMEC does not allow for radial magnetic perturbations B[s].



The Hessian matrix

Total energy for a stationary plasma (magnetic + internal):

$$W = \int \left[\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1}\right] dVol$$

For quadratic energies, the Hessian is essentially the coefficient of the second-order term (with a negative sign due to the conventions used in the code). For a spring potential, the Hessian would essentially be the spring constant.

$$W = \frac{1}{2}k\xi^2$$

$$\vec{H} = -\frac{\partial^2 W}{\partial\xi^2} = -k\vec{I}$$

The Hessian matrix used in SIESTA is defined below. The eigenspectrum for this matrix will be discussed throughout the poster for both VMEC and SIESTA equilibria.

$$\vec{H} = \frac{\partial\vec{F}}{\partial\vec{\xi}} = -\frac{\partial^2 W}{\partial\vec{\xi}^2}$$

This Hessian is negative definite (has only negative eigenvalues) for a completely stable equilibrium.

$$\vec{H}\vec{\xi} = -\omega^2\vec{\xi}$$

The SIESTA nullspace

The nullspace of the Hessian matrix in SIESTA is very important because it can lead to huge displacements in directions that result in no change to the MHD force. It can easily be seen from the linearized ideal MHD equations that a plasma displacement that is purely parallel to the magnetic field everywhere will result in zero contribution to the linear force.

$$\delta\vec{F} = \delta\vec{J} \times \vec{B}_0 + \vec{J}_0 \times \delta\vec{B} - \nabla\delta p$$

$$\delta\vec{B} = \nabla \times \left(\vec{\xi}_{//} \times \vec{B}_0\right) = 0$$

$$\delta\vec{J} = \frac{1}{\mu_0}\left(\nabla \times \delta\vec{B}\right) = 0$$

$$\delta p = (\gamma - 1)\vec{\xi}_{//} \cdot \nabla p_0 - \gamma\nabla \cdot \left(p_0\vec{\xi}_{//}\right) = -\gamma p_0 \nabla \cdot \vec{\xi}_{//} \approx 0$$

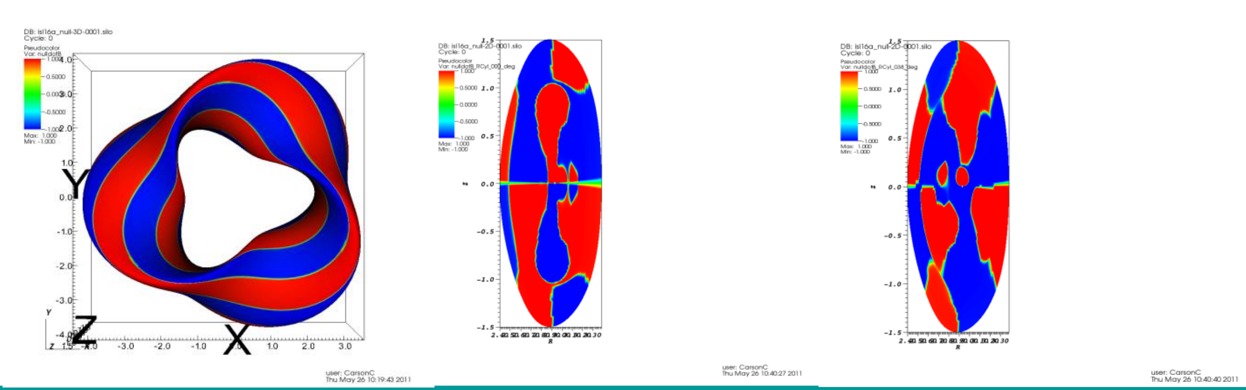
$$\delta\vec{F} = \delta\vec{J} \times \vec{B}_0 + \vec{J}_0 \times \delta\vec{B} - \nabla\delta p \approx 0$$

Thus we would expect that a parallel plasma displacement would serve as a nullspace vector for the Hessian matrix; that is a parallel displacement should satisfy the following equation:

$$\vec{H}\vec{\xi}_{//} = \frac{\partial\vec{F}}{\partial\vec{\xi}}\vec{\xi}_{//} = \delta\vec{F} = 0$$

Put another way, we hope to find numerically that the nullspace eigenvectors of the SIESTA Hessian matrix are displacements that are essentially parallel to the magnetic field everywhere in the domain.

Below are some results for the computed nullspace structure for a three-field period stellarator. A plasma displacement purely in the direction of this mode would result in essentially no change to the linearized MHD force. These plots of the nullvector dotted into the magnetic field show that the mode is parallel (red) or antiparallel (blue) to the magnetic field everywhere, as expected. There is a large degeneracy of eigenmodes that result in no net force.

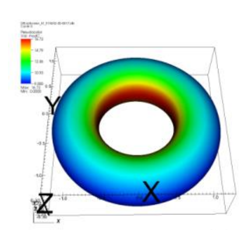


Configuration convergence

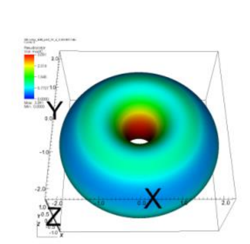
Tokamaks are in general much easier to converge to a stable equilibrium. Strongly shaped stellarators like HSX are more difficult due to larger mode content and a more complicated Hessian matrix. This matrix will generally have a larger condition number, making the linear solve at each step much more challenging. Convergence work on SIESTA is ongoing to improve performance on helical stellarators.

Tokamak convergence

Solov'ev
Initial run force residual: 9.518E-13

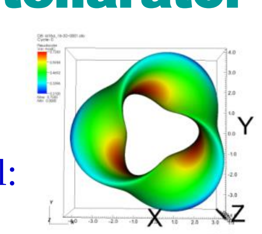


CDX-U
Initial run force residual: 1.010E-30

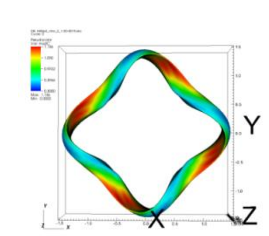


Stellarator convergence

3 field-period stellarator
Initial run force residual: 5.039E-32



HSX
Initial run force residual: 1.157E-01



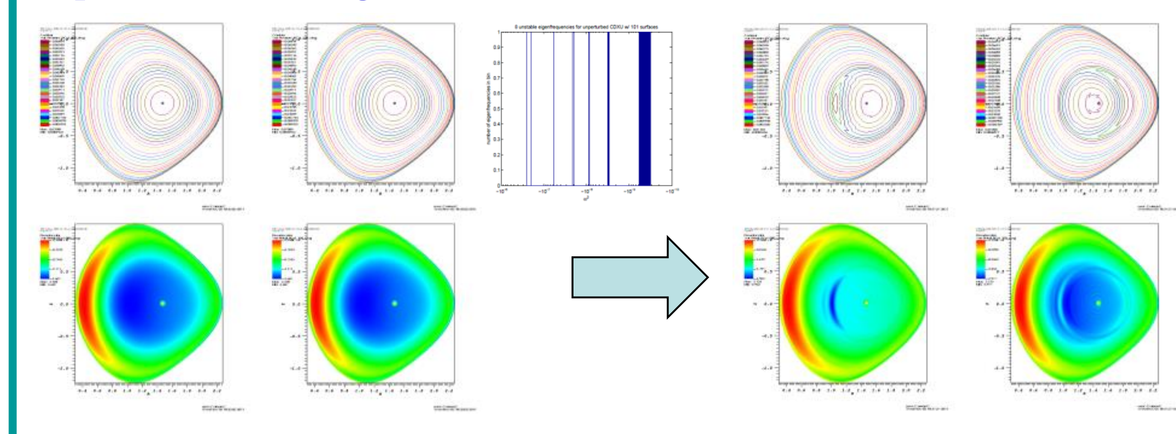
Take-away: The stronger-shaped, far-from-axisymmetric stellarators are more difficult to converge due to their larger mode structure.

High beta CDX-U stability analysis

A stability analysis was performed on a high beta (8%) CDX-U equilibrium that is known to be Mercier unstable. CDX-U (Current Drive Experiment-Upgrade) was a small tokamak located at Princeton Plasma Physics Lab.

The LAPACK subroutine dgeev was used to solve for all of the eigenvalues of the SIESTA Hessian matrix. Due to conventions used in the code, negative eigenvalues correspond to stable modes and positive eigenvalues correspond to unstable modes.

CDX-U axisymmetric VMEC equilibrium converged on SIESTA mesh | **CDX-U SIESTA equilibrium w/ m=1, n=1 island**



Why are there still unstable eigenmodes appearing in this converged equilibrium?

During cross-checks, it was determined that dgeev does NOT solve for the smaller eigenvalues of our Hessian to any reasonable precision. The calculated "unstable" modes were shown to yield a positive δW. This demonstrates that the sign of the small dgeev-calculated eigenvalues could not be trusted, explaining the appearance of spurious unstable modes.

$$dW = \langle x|H|x\rangle = -\int dVol\, \vec{x}^T\vec{F} \approx -\int dVol\, \vec{x}^T H\vec{x}$$

For a stable eigenmode, this leads to the following formula: $dW = -\lambda\int dVol\, \vec{x}^T\vec{x} > 0$

Since the change in energy was positive, the positive eigenvalues corresponding to unstable modes were calculated inaccurately by dgeev and should actually be negative, stable eigenvalues.

The table below demonstrates that dgeev does NOT solve the eigenvalues (notably the smaller ones) to any reasonable precision for matrices of condition number ~10^{12}.

Config	Condition number	Smallest eigenvalue	Largest eigenvalue	||Hx-lx||/||lx|| (-25.926)	||Hx-lx||/||lx|| (-5.179E-002)	||Hx-lx||/||lx|| (4.449E-007)
CDX-U	7.286E11	-8.282E-010	-17.702	6.970E-004	1.8126E-002	468.390
	Condition number	Smallest eigenvalue	Largest eigenvalue	||Hx-lx||/||lx|| (-79.082)	||Hx-lx||/||lx|| (2.179E-004)	||Hx-lx||/||lx|| (-7.687E-008)
Solov' ev	7.416E11	-7.687E-008	-15448.345	2.865E-003	320.469	634112.128

SLEPc is being implemented so that the eigenspectrum can be solved in an accurate manner. SLEPc is highly scalable and includes multiple direct and iterative solvers.

Summary and future work

•The nullspace eigenmodes of the SIESTA Hessian matrix have successfully been demonstrated to be parallel plasma displacements, agreeing with theory.

•A stability analysis has been performed on CDX-U, demonstrating the presence of unstable modes in an axisymmetric VMEC equilibrium. The appearance of unstable modes in converged SIESTA equilibria brought into question the eigensolver, dgeev.

•dgeev is insufficient to resolve the small eigenvalues necessary for an accurate stability analysis. SLEPc will be used next as it should be able to handle eigenvalues of vastly varying magnitudes.

•The SIESTA eigenspectrum work will be useful for future work on Alfven eigenspectrum analysis in the presence of islands. 1D, 2D, and 3D equilibria will be analyzed.

References

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