Stability analysis and non-field-periodic islands with the SIESTA equilibrium code

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Overview

○SIESTA (Scalable Iterative Equilibrium Solver for Toroidal Applications) is a 3D nonlinear MHD equilibrium solver capable of resolving islands in confinement devices in an accurate and scalable manner.

○A stability analysis has been carried out for the CDX-U tokamak and a Solov'ev tokamak configuration. Problems were detected with the LAPACK eigensolver dgeev when used on our poorly-conditioned Hessian matrix. As a result, SLEPc is currently being studied.

○The eigenspectrum work is an important stepping stone for future work on the physical effect of islands on the MHD Alfven spectrum. The impact of islands on the presence of gap modes and the existence of various Alfven modes in 3D equilibria will be studied in future work.

○A non-field-periodic version of SIESTA is being developed to allow for perturbations and islands with full device structure. These islands lacking field-periodicity are seen in some stellarators such as CTH, which will be discussed here. The new code will be able to resolve these structures.

Basic equations of SIESTA

The force in the current iteration's unperturbed state which is not necessarily yet in equilibrium is given by:

$$\vec{F}_0 = \vec{J}_0 \times \vec{B}_0 - \nabla p_0$$

For any plasma displacement, the linearized system becomes the following:

Perturbed, linear system $\delta\vec{F} = \delta\vec{J} \times \vec{B}_0 + \vec{J}_0 \times \delta\vec{B} - \nabla \delta p$

Faraday's law $\delta\vec{B} = \nabla \times \left(\vec{\xi} \times \vec{B}_0\right)$

Ampere's law $\delta\vec{J} = \frac{1}{\mu_0}\left(\nabla \times \delta\vec{B}\right)$

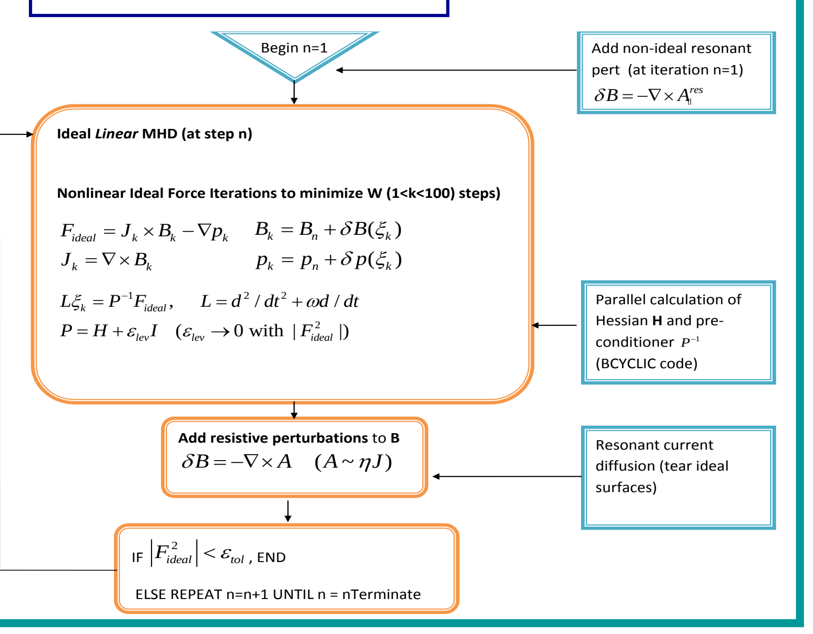
Mass conservation $\delta p = (\gamma - 1)\vec{\xi} \cdot \nabla p_0 - \gamma\nabla \cdot \left(p_0 \vec{\xi}\right)$

To find equilibrium, minimize nonlinear force $\boxed{\vec{F} \equiv \vec{J} \times \vec{B} - \nabla p \to \vec{0}}$ and update fields.

Linear equation to be solved iteratively by GMRES:

$$\vec{H}\vec{\xi} = -\vec{F}_{res} \qquad \frac{\partial \vec{F}}{\partial \vec{\xi}}\vec{\xi} = -\vec{F}_{res}$$

At a VMEC or SIESTA equilibrium, the eigenvalues of the SIESTA Hessian matrix correspond to actual physical stable/unstable modes. VMEC equilibria will in general contain unstable modes in the SIESTA context, because VMEC does not allow for radial magnetic perturbations B^s.



The Hessian matrix

Total energy for a stationary plasma (magnetic + internal):

$$W = \int \left[\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1}\right] dVol$$

For quadratic energies, the Hessian is essentially the coefficient of the second-order term (with a negative sign due to the conventions used in the code). For a spring-like potential, the Hessian would basically be the spring constant.

$$W = \frac{1}{2}k\xi^2$$

$$\vec{H} = -\frac{\partial^2 W}{\partial \xi^2} = -k\vec{I}$$

The Hessian matrix used in SIESTA is defined below. The eigenspectrum for this matrix will be discussed throughout the poster for both VMEC and SIESTA equilibria. Due to the disparity of MHD scales, the Hessian matrix is very ill-conditioned (CN typically on the order of 10^{12}).

$$\vec{H} = \frac{\partial \vec{F}}{\partial \vec{\xi}} = -\frac{\partial^2 W}{\partial \vec{\xi}^2}$$

This Hessian is negative definite (has only negative eigenvalues) for a completely stable equilibrium.

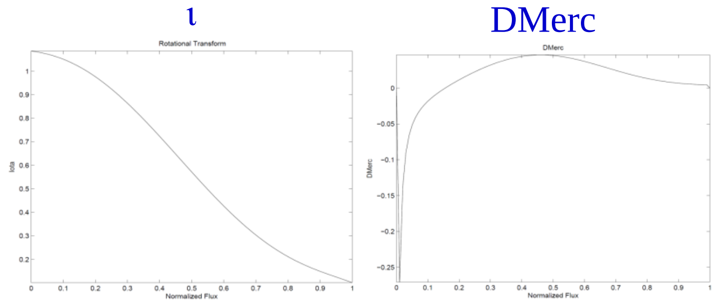
Eigenproblem:

$$\vec{H}\vec{\xi} = -\omega^2 \vec{\xi}$$

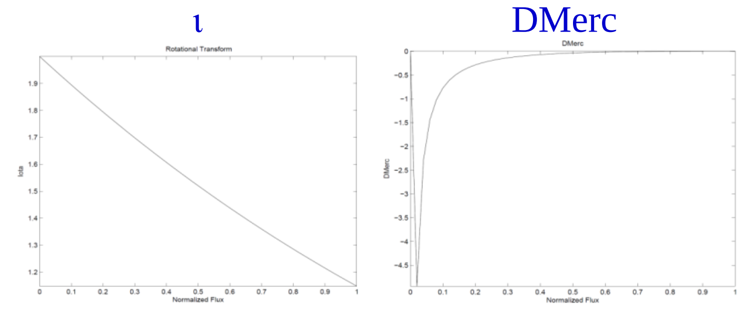
High beta CDX-U and Solov'ev stability analysis

A stability analysis was performed on a high beta (8%) CDX-U equilibrium that is known to be Mercier unstable. CDX-U (Current Drive Experiment-Upgrade) was a small tokamak located at Princeton Plasma Physics Lab. In addition, a Solov'ev circular tokamak equilibrium was studied.

CDX-U 8%

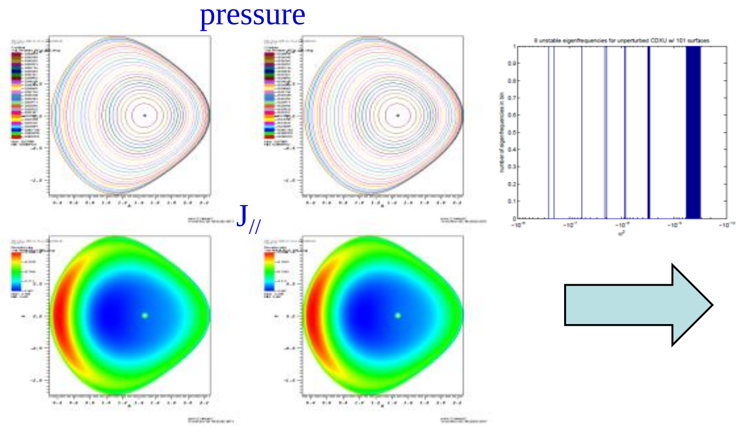


Solov'ev tokamak

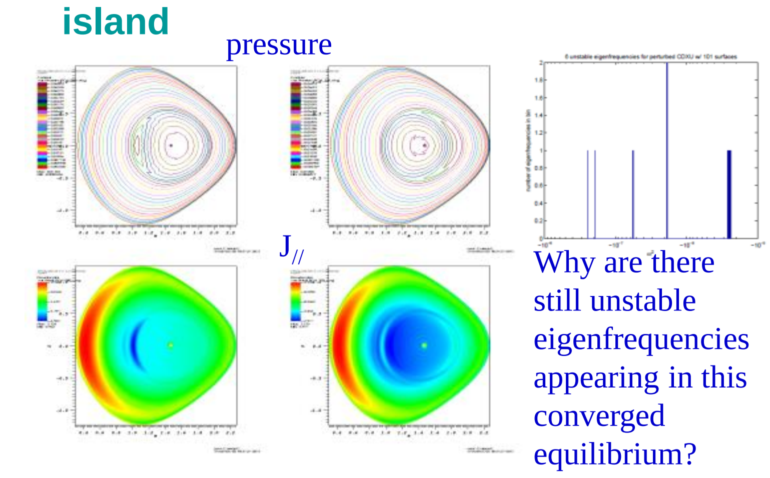


Mercier's criterion for stability, $D_{Merc} > -\frac{1}{4}$, is clearly violated in both cases

CDX-U axisymmetric VMEC equilibrium converged on SIESTA mesh



CDX-U SIESTA equilibrium w/ m=1, n=1 island



Why are there still unstable eigenfrequencies appearing in this converged equilibrium?

The LAPACK subroutine dgeev was used to solve for all of the eigenvalues of the SIESTA Hessian matrix. Due to conventions used in the code, negative eigenvalues correspond to stable modes and positive eigenvalues correspond to unstable modes.

During cross-checks, it was determined that dgeev does NOT solve for the smaller eigenvalues of our Hessian to any reasonable precision. The calculated "unstable" modes were shown to actually yield a positive δW. This demonstrates that the sign of the small dgeev-calculated eigenvalues can not be trusted, explaining the appearance of spurious unstable modes.

$$\delta W = \langle x|H|x\rangle = -\int dVol\, \vec{x}^T\vec{F} \approx -\int dVol\, \vec{x}^T H \vec{x}$$

For a stable eigenmode, this leads to the following formula:

$$\delta W = -\lambda \int dVol\, \vec{x}^T\vec{x} > 0$$

Since the change in energy was positive for the "unstable" modes, the small, positive eigenvalues were calculated inaccurately by dgeev and should actually be negative, stable eigenvalues.
The table below demonstrates that dgeev does NOT solve the eigenvalues (most notably the smaller ones) to any reasonable precision for matrices of condition number ~10^{12}. The SIESTA simulations used had ns=51, mpol=6, and ntor=2, giving a 5355 x 5355 Hessian matrix.

Config	Condition number	Smallest eigenvalue	Largest eigenvalue	||Hx-lx||/||lx|| (-25.926)	||Hx-lx||/||lx|| (-5.179×10^{-2})	||Hx-lx||/||lx|| (4.449×10^{-7})
CDX-U	7.286×10^{11}	-8.282×10^{-10}	-17.702	6.970×10^{-4}	1.8126×10^{-2}	468.390
	Condition number	Smallest eigenvalue	Largest eigenvalue	||Hx-lx||/||lx|| (-79.082)	||Hx-lx||/||lx|| (2.179×10^{-4})	||Hx-lx||/||lx|| (-7.687×10^{-8})
Solov'ev	7.416×10^{11}	-7.687×10^{-8}	-1.5458×10^{4}	2.865×10^{-3}	320.469	6.341×10^{5}

To remedy the problems with dgeev, SLEPc is being implemented so that the eigenspectrum can be solved in an accurate manner. It is highly scalable and includes multiple iterative solver options.

SLEPc allows for control of many parameters for the eigenproblem

- •Residual tolerance
- •Number of iterations
- •Number of desired eigenvalues
- •Target range of eigenvalues to be computed
- •Iterative eigensolver to be used (Lanczos, Arnoldi, Krylov-Schur, Davidson…)

Removal of field-periodicity requirement is nearing completion

Current version of the code only includes toroidal mode numbers that are multiples of the number of field periods.

$$\xi^\rho(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(\xi^\rho)_{mn}(\rho)\cos(m\theta + nN_{fp}\phi)$$
$$\xi^\alpha(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(\xi^\alpha)_{mn}(\rho)\sin(m\theta + nN_{fp}\phi), \alpha \in \{\theta,\phi\}$$
$$B^\rho(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(B^\rho)_{mn}(\rho)\sin(m\theta + nN_{fp}\phi)$$
$$B^\alpha(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(B^\alpha)_{mn}(\rho)\sin(m\theta + nN_{fp}\phi), \alpha \in \{\theta,\phi\}$$
$$p(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}p_{mn}(\rho)\cos(m\theta + nN_{fp}\phi)$$

Include all toroidal modes →

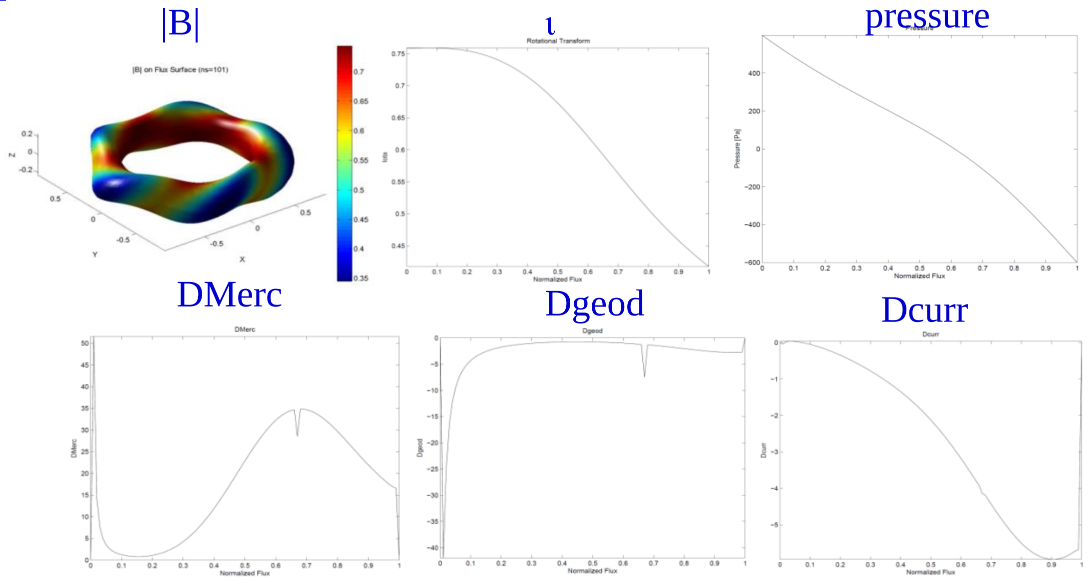
Only a single field period of the device needs to be modeled when field-periodicity is enforced.

$$\xi^\rho(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(\xi^\rho)_{mn}(\rho)\cos(m\theta + n\phi)$$
$$\xi^\alpha(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(\xi^\alpha)_{mn}(\rho)\sin(m\theta + n\phi), \alpha \in \{\theta,\phi\}$$
$$B^\rho(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(B^\rho)_{mn}(\rho)\sin(m\theta + n\phi)$$
$$B^\alpha(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}(B^\alpha)_{mn}(\rho)\sin(m\theta + n\phi), \alpha \in \{\theta,\phi\}$$
$$p(\rho,\theta,\phi) = \sum_{m=0}^{mpol}\sum_{n=-ntor}^{ntor}p_{mn}(\rho)\cos(m\theta + n\phi)$$

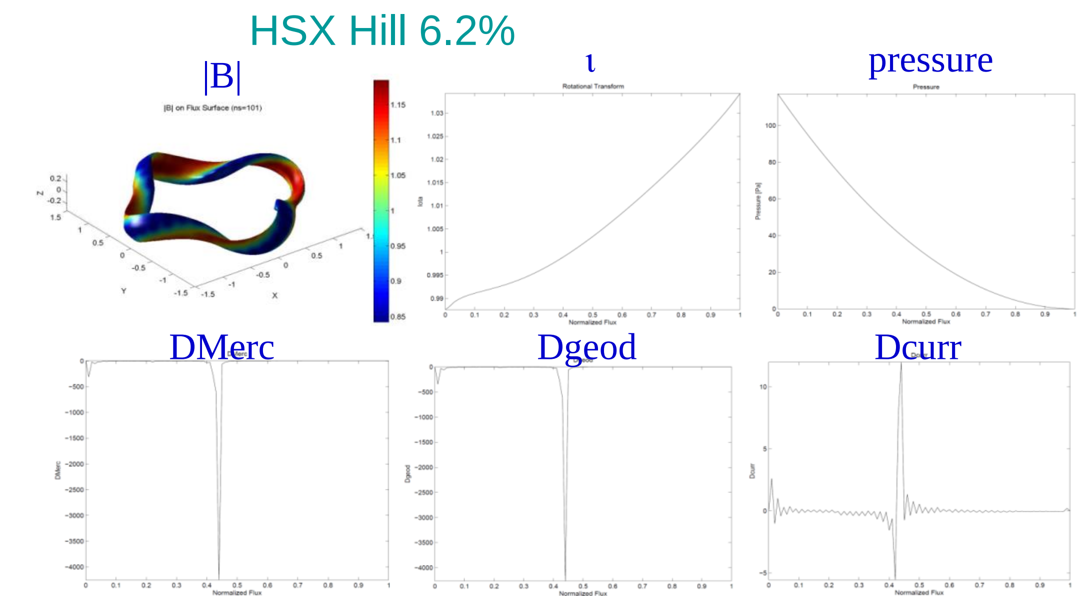
CTH (Compact Toroidal Hybrid) example equilibrium

CTH is a stellarator with current drive located at Auburn University. It will serve as a useful test case for this modification of the code because it is unstable to island formation with lower toroidal mode number n than the number of field periods (5). This particular equilibrium with a plasma current is unstable to an n=1 island formation.

Rotational transform profile shows ι=1/2 surface present around r/a=0.82. This should be unstable to m=2, n=1 non-field-periodic island formation. The stability parameters for a standard field-periodic VMEC run show no sign of instability. This is because n=1 is not included in the spectral content of the code. Only n values which are a multiple of the number of field periods (5) are included. An HSX case with ι=4/4 is unstable to field-periodic (n=4) island formation. Thus the instability is clearly visible. This behavior should be seen in the CDX-U case after removing the field-periodic restriction.



HSX Hill 6.2%



Summary and future work

•A stability analysis has been performed on CDX-U and a Solov'ev tokamak, demonstrating the presence of unstable modes in axisymmetric VMEC equilibria. The appearance of unstable modes in converged SIESTA equilibria brought into question the dgeev eigensolver.
•dgeev is unable to resolve the small eigenvalues necessary for an accurate stability analysis. SLEPc is being implemented as it is able to handle eigenvalues of vastly varying magnitudes through many different iterative methods.
•The eigenspectrum analysis will be used as a precursor to an Alfven spectrum analysis.
•SIESTA is being modified to include non-field-periodic islands. This will initially be used to study n=1 islands in the CTH stellarator.

References

[1] S. P. Hirshman, R. Sanchez, and C. R. Cook, "SIESTA: a scalable iterative equilibrium solver for toroidal applications," Physics of Plasmas, vol. 18, no. 6, pp. 062504-1-062504-13, 2011.
[2] S. P. Hirshman and J. C. Whitson, "Steepest-descent moment method for three-dimensional magnetohydrodynamic equilibria," Physics of Fluids, vol. 26, no. 12, pp. 3553-3568, 1983.
[3] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, "An energy principle for hydromagnetic stability problems," Proceedings of the Royal Society of London Series A. Mathematical and Physical Sciences, vol. 244, no. 1236, pp. 17-40, 1958.
[4] S. P. Hirshman, K. S. Perumalla, V. E. Lynch, and R. Sanchez, "BCYCLIC: a parallel block tridiagonal matrix cyclic solver," Journal of Computational Physics, vol. 229, no. 18, pp. 6392-6404, 2010.

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