The shear Alfven continuum within the separatrix of a magnetic island

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1. Motivation and overview

• The shear Alfven spectrum of a magnetically confined plasma influences the stability properties of the system.
• Discrete Alfven eigenmodes are of particular interest to fusion plasmas. These modes exist in gaps of the Alfven continuum, and do not experience continuum damping.
• Recent results by Koliner et al. showed the existence of Alfven modes in MST plasmas. The results have not been fully explained as a TAE mode, and magnetic islands may play a role.
• Previous work has shown that the presence of a magnetic island can produce additional gaps in the Alfven spectrum (Biancalani et al. [2011]). This work used a simplified straight flux tube slab model approximation for the island structure and assumed eigenmodes with the same helicity as the island.
• A new theory has been developed to describe the Alfven spectrum in a cylindrical plasma containing a magnetic island.
• Approximate analytical solutions near the O-point have been computed, which agree with results of Biancalani’s model.
• Initial numerical results have been computed near the separatix and near the O-point, showing qualitative agreement.

2. Island coordinate system

A system of coordinates describing a cylindrical plasma with a magnetic island can be defined in terms of the cylindrical coordinates \((\rho, \zeta, \phi = 2\pi z/L)\) of the equilibrium background magnetic field \(B_0\) where \(L\) is the length of the cylinder:

\[
B = B_0 + B_1 = B_0 \hat{B}_0 + \rho \hat{B}_1 + \zeta \hat{B}_2
\]

A set of helical island coordinates a la Hegna and Callen [1992] can be defined as:

\[
\Psi^* = \frac{\rho}{\theta_0} \cos(\alpha)
\]

\[
\Phi = \frac{\rho}{\theta_0} \frac{\partial \phi}{\partial \rho}
\]

\[
x = \rho - \rho_0
\]

\[
z = \frac{\zeta - \zeta_0}{\theta_0}
\]

\[
\alpha = \frac{\zeta - \zeta_0}{\theta_0} + \phi_0
\]

\[
B = B_0 + B_1 = B_0 \hat{B}_0 + \rho \hat{B}_1 + \zeta \hat{B}_2
\]

This is a Sturmt-Liouville problem with coupled boundary conditions. Next we want to study the behavior of the solution in two asymptotic regimes inside the magnetic island near the O-point and just inside the separatix.

3. Shear Alfven continua in presence of a magnetic island

The linearized ideal MHD equations are the momentum equation, the combined Faraday’s law/Ohms’s law, and the equation of state:

\[
p \omega^2 \zeta = \nabla \phi + \delta B \times J + B \times (\nabla \times \delta B) = \delta B = \nabla \times (\zeta \times B).
\]

This equation can be simplified for the background cylinder case of interest and by considering zero beta plasma \(P = 0\). Looking for periodic solutions with non-square-integrable radial singularity results in the condition from Cheng and Chance [1986]:

\[
B \cdot \nabla (|\nabla \phi|^2 - B_0^2) + \rho_1 \nabla (|\nabla \phi|^2 - B_0^2) = 0.
\]

Under this assumption, the \(B \cdot \nabla \) operator can be expressed as

\[
B \nabla \phi^* = \frac{B_0}{\rho_0} \left( \frac{\alpha}{\rho} - \frac{\partial}{\partial \rho} \right) \phi^* = \frac{B_0}{\rho_0} \left( \frac{\alpha}{\rho} - \frac{\partial}{\partial \rho} \right) \phi^* = \left( \frac{\alpha}{\rho} - \frac{\partial}{\partial \rho} \right) \phi^*.
\]

Thus the differential equation can be written as a second-order ODE in \(\alpha^*\) for each flux surface \(\Psi^*\) (using \(\alpha_2 = 2\pi \psi_{xq}/q_2\)):

\[
\frac{d}{d\alpha^*} \left( \frac{d}{d\alpha^*} \alpha^* \right) + \omega^2 \alpha^* = 0.
\]

\[
Y = \nu_1(\alpha^*) e^{\omega \alpha^*}, \quad \omega = 2 \arcsin \left( \frac{\pi}{2} \left( \frac{2K(\alpha^*)}{\alpha^*} \right) \right).
\]

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4. Analytic behavior near the O-point

Under the assumption that \(\epsilon > 0\), the O-point is located at \(\alpha = 0\) and \(\Psi^* = -A\). On a flux surface very near the O-point, \(\Psi^* / A \approx -1\) and \(\epsilon \ll 1\). This allows us to approximate \(x\) as

\[
x = \frac{\epsilon}{q_0} \left( \frac{\zeta - \zeta_0}{\theta_0} + \phi_0 \right) \approx \frac{\epsilon}{q_0} \left( \frac{\zeta - \zeta_0}{\theta_0} + \phi_0 \right) \approx \frac{\epsilon}{q_0} \left( \frac{\zeta - \zeta_0}{\theta_0} + \phi_0 \right)
\]

This simplifies the differential equation to

\[
\frac{d^2Y}{dx^2} + \frac{\omega^2}{q_0^2} Y = 0.
\]

This differential equation describes a simple harmonic oscillator, and periodicity must be enforced:

\[
Y = Y_0 e^{j \omega t}
\]

Finally, the Alfven frequencies in the vicinity of the O-point are given by the following, making use of the fact that \(\Omega \approx \omega_0 q_4 / 4\):

\[
\omega^2 = \frac{1}{16} \left( \frac{q_4}{q_0} \right)^2 \omega_0^2
\]

This matches the result of Biancalani et al. which is given by

\[
\omega^2 = \frac{1}{4} \omega_0^2
\]

This agrees with the result from Biancalani et al. [2011]. Both models demonstrate an \(\omega^2 \approx \omega_0^2\) dependence on the island size.

5. Numerical solution of Alfven eigenmode equation

The Sturm-Liouville problem can be converted into a Schrodinger’s equation and the potential can be analyzed. Near the separatrix, the potential looks like a periodic delta-function double well. As expected, the eigenmodes are found localized in the well.

6. Conclusions

- The shear Alfven spectrum near the O-point of the island in a cylindrical plasma has been approximated analytically using a set of straight-line magnetic field coordinates.
- The results match those obtained by Biancalani et al. [2011].
- The analytical solution appears to hold for modes with the same helicity as the magnetic island near the O-point and just inside the separatrix. The trend of the lowest eigenfrequencies appears to follow Biancalani’s result qualitatively.
- The continuous spectrum shows a frequency gap present near the O-point of the magnetic island. The possibility for a discrete Alfven eigenmode to exist in this gap will be studied in the future.
- The eigenspectrum will be solved throughout the domain inside and outside the island and for general \(\epsilon \neq 0\) as future work.

References


