The shear Alfvén continuum within the separatrix of a magnetic island

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1. Motivation and overview

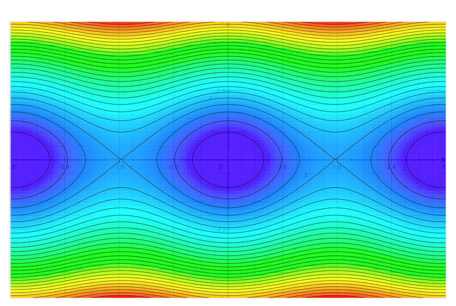
- The shear Alfven spectrum of a magnetically confined plasma influences the stability properties of the system.
- Discrete Alfven eigenmodes are of particular interest to fusion plasmas. These modes exist in gaps of the Alfven continuum, and do not experience continuum damping.
- Recent results by Koliner et al. showed the existence of Alfven modes in MST plasmas. The results have not been fully explained as a TAE mode, and magnetic islands may play a role.
- Previous work has shown that the presence of a magnetic island can produce additional gaps in the Alfven spectrum (Biancalani et al. [2011]). This work used a simplified straight flux tube slab model approximation for the island structure and assumed eigenmodes with the same helicity as the island.
- A new theory has been developed to describe the Alfven spectrum in a cylindrical plasma containing a magnetic island.
- Approximate analytical solutions near the O-point have been computed, which agree with the results of Biancalani's model.
- Initial numerical results have been computed near the separatrix and near the O-point, showing qualitative agreement.

2. Island coordinate system

A system of coordinates describing a cylindrical plasma with a magnetic island can be defined in terms of the cylindrical coordinates $(r, \theta, \zeta = 2\pi z/L)$ of the equilibrium background magnetic field $\mathbf{B_0}$ where L is the length of the cylinder:

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} = B_\theta \hat{\theta} + B_z \hat{z} + \mathbf{B_1}$$

$$\frac{r_0}{B_\theta} \mathbf{B_1} \cdot \hat{r} = A \sin(m_0\theta - n_0\zeta - \phi_0)$$



A set of helical island coordinates a la Hegna and Callen [1992] can be defined as:

$$\Psi^* \approx q_0' \frac{x^2}{2} - A\cos(\alpha) \qquad \kappa^2 = \frac{\Psi^* + A}{2A}$$

$$\Phi^* = \pm \frac{wE(k)}{\pi k} \qquad \Omega = \frac{d\Psi^*}{d\Phi^*} = \frac{\pi q_0' w}{8K(\kappa^2)}$$

$$x = r - r_0 \qquad \alpha^* = \frac{\pi}{2K(\kappa)} F\left(\arcsin\left[\frac{1}{\kappa}\sin\left(\frac{\alpha}{2}\right)\right], \kappa^2\right)$$

$$\chi = \frac{\theta - \zeta}{1 - q_0}$$

$$\alpha = \zeta - q_0\theta + \phi_0$$

In this island coordinate system, the total magnetic field can be written in a straight field-line coordinate form as

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} = \nabla\alpha^* \times \nabla\Phi^* + \Omega\nabla\Phi^* \times \nabla\chi,$$

References

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3. Shear Alfvén continua in presence of a magnetic island

The linearized ideal MHD equations are the momentum equation, the combined Faraday's law/Ohm's law, and the equation of state:

$$\rho\omega^2\xi = \nabla\delta p + \delta\mathbf{B} \times \mathbf{J} + \mathbf{B} \times (\nabla \times \delta\mathbf{B}) \qquad \delta\mathbf{B} = \nabla \times (\xi \times \mathbf{B}).$$

This equation can be simplified for the background cylinder case of interest and by considering zero beta plasmas $P = 0$. Looking for periodic solutions with non-square-integrable radial singularity results in the condition from Cheng and Chance [1986]:

$$\mathbf{B} \cdot \nabla\left(\frac{|\nabla\Phi^*|^2}{B^2} \mathbf{B} \cdot \nabla\xi_s\right) + \rho\omega^2 \frac{|\nabla\Phi^*|^2}{B^2}\xi_s = 0.$$

The surface displacement will be assumed to be of the following form:

$$\xi_s(\chi, \alpha^*) = \xi_0(\alpha^*) e^{-il\chi}.$$

Under this assumption, the $\mathbf{B} \cdot \nabla$ operator can be expressed as

$$\mathbf{B} \cdot \nabla\xi_s = \frac{B_\theta}{r_0}\left(\Omega\frac{\partial}{\partial\alpha^*} - il\right)\xi_s = \frac{B_\theta}{r_0} e^{\frac{il}{\Omega}\alpha^*} \Omega \frac{\partial}{\partial\alpha^*}\left(\xi_s e^{-\frac{il}{\Omega}\alpha^*}\right).$$

Thus the differential equation can be written as a second-order ODE in α^* for each flux surface Ψ^* (using $\omega_A = 2\pi v_A / q_0 L$):

$$\frac{d}{d\alpha^*}\left(x^2 \frac{d}{d\alpha^*} Y\right) + \frac{\omega^2 x^2}{\omega_A^2 \Omega^2} Y = 0, \qquad Y = \xi_0(\alpha^*) e^{-\frac{il}{\Omega}\alpha^*}$$

$$x^2 = \frac{2}{q_0'}(\Psi^* + A\cos\alpha) \qquad \alpha = 2\arcsin\left(\kappa sn\left[\frac{2K(\kappa^2)\alpha^*}{\pi}, \kappa^2\right]\right)$$

This is a Sturm-Liouville problem with coupled boundary conditions. Next we want to study the behavior of the solution in two asymptotic regimes inside the magnetic island: near the O-point and just inside the separatrix.

4. Analytic behavior near the O-point

Under the assumption that $q_0' > 0$, the O-point is located at $\alpha = 0$ and $\Psi^* = -A$. On a flux surface very near the O-point, $\Psi^*/A \approx -1$ and $\alpha_{max} = \varepsilon$ where $\varepsilon << 1$. This allows us to approximate x as

$$x^2 = \frac{2A}{q_0'}(-1 + \cos\varepsilon) \approx \frac{A\varepsilon^2}{q_0'}$$

This simplifies the differential equation to

$$\frac{d^2Y}{d\alpha^{*2}} + \frac{\omega^2}{\omega_A^2\Omega^2} Y = 0$$

This differential equation describes a simple harmonic oscillator, and periodicity must be enforced:

$$Y = Y_0 e^{i\frac{\omega}{\omega_A\Omega}\alpha^*} = Y_0 e^{ij\alpha^*}$$

This gives us our condition for solution as

$$\frac{\omega}{\omega_A\Omega} = j$$

Finally, the Alfven frequencies in the vicinity of the O-point are given by the following, making use of the fact that $\Omega \approx q_0' w/4$:

$$\omega^2 = j^2\Omega^2\omega_A^2 = \frac{1}{16} j^2 (q_0' w)^2 \omega_A^2$$

This matches the result of Biancalani et al which is given by

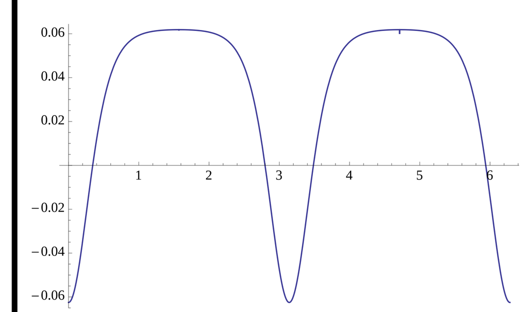
$$\omega^2 = \frac{j^2\omega_A^2}{q_{in}^2}$$

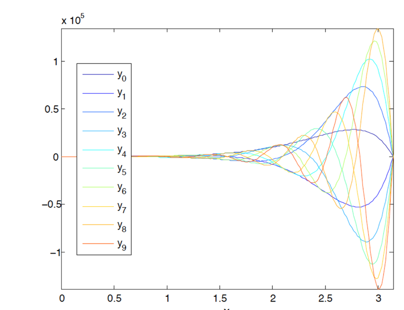
where $q_{in} = 1/\Omega$.

This agrees with the result from Biancalani et al. [2011]. Both models demonstrate an $\omega^2 \sim A$ dependence on the island size.

5. Numerical solution of Alfvén eigenmode equation

The Sturm-Liouville problem can be converted into a Schrodinger's equation and the potential can be analyzed. Near the separatrix, the potential looks like a periodic delta-function double well. As expected, the eigenmodes are found localized in the well.





The Alfven eigenmode equation

$$\frac{d}{d\alpha^*}\left(x^2 \frac{d}{d\alpha^*} Y\right) + \frac{\omega^2 x^2}{\omega_A^2 \Omega^2} Y = 0,$$

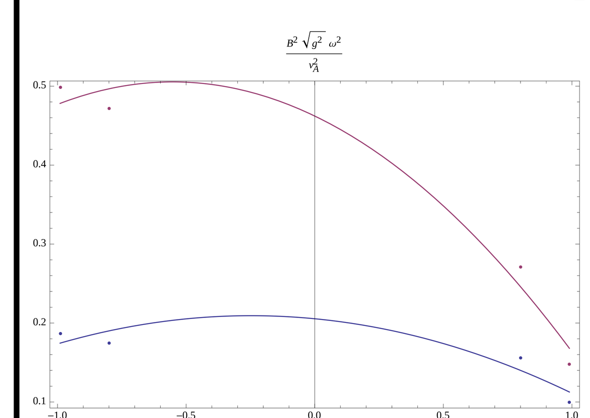
is a Sturm-Liouville problem with

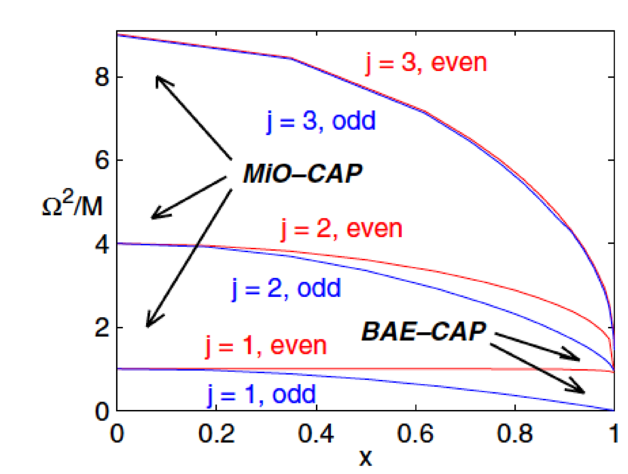
$$p(\alpha^*) = x^2 \qquad w(\alpha^*) = \frac{x^2}{\Omega^2}$$

Taking $l = 0$, that is restricting our eigenmodes to the same helicity of the island results in periodic boundary conditions:

$$Y(\alpha^* = 0) = Y(\alpha^* = 2\pi) \qquad Y'(\alpha^* = 0) = Y'(\alpha^* = 2\pi)$$

Since $p(\alpha^*)$ and $w(\alpha^*)$ are periodic, the domain can be cut in half and this reduces to four problems with separated boundary conditions. These simplified problems can be solved with the MATLAB code MATSLISE. The lowest eigenfrequencies were computed near the O-point and near the separatrix, since close to the O-point $sn\left[\frac{2K(\kappa^2)\alpha^*}{\pi}, \kappa^2\right] \approx \sin\left[\frac{2K(\kappa^2)\alpha^*}{\pi}\right]$ and close to the separatrix $sn\left[\frac{2K(\kappa^2)\alpha^*}{\pi}, \kappa^2\right] \approx \tanh\left[\frac{2K(\kappa^2)\alpha^*}{\pi}\right]$.





Biancalani et al. [2011]

Qualitatively, these two models appear to agree. Notice the existence of a gap near the O-point. This implies that discrete Alfven eigenmodes can exist near the magnetic island O-point that will not experience continuum damping.

6. Conclusions

- The shear Alfven spectrum near the O-point of the island in a cylindrical plasma has been approximated analytically using a set of straight-line magnetic field coordinates. The result matches that obtained by Biancalani et al. [2011].
- The problem has been solved numerically for modes with the same helicity as the magnetic island near the O-point and just inside the separatrix. The trend of the lowest eigenfrequencies appears to match Biancalani's result qualitatively
- The continuous spectrum shows a frequency gap present near the O-point of the magnetic island. The possibility for a discrete Alfven eigenmode to exist in this gap will be studied in the future.
- The eigenspectrum will be solved throughout the domain inside and outside the island and for general $l \neq 0$ as future work.