Theory and simulation of the magnetic island-modified BAE gap

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Outline

- Preliminaries on Alfven spectrum
- Observed Alfvenic activity in MST
- TAE and BAE/BAAE description insufficient
- Modifications to Alfven spectrum due to magnetic island
- Demonstrate that an island-shifted BAE gap is consistent with experimental observations
- Proposed work: using SIESTA equilibrium to investigate Alfven spectrum with island

Preliminaries: shear Alfvén waves

Shear (or torsional) Alfven waves are incompressible waves that propagate along the direction of the magentic field lines.

- $v_A = \frac{B}{\sqrt{\mu_0 \rho}}$
- $\omega = v_A k_{\parallel}$
- Perturbed electric field and fluid velocity are perpendicular to the magnetic field.
  - "Plucking" the magnetic field line

Alfvén continuum arises from the linearized ideal MHD model

The linearized ideal MHD equations are the momentum equation, the combined Faraday's law/Ohm's law, and the equation of state:

$$\rho\omega^2\xi = \nabla\delta p + \delta\mathbf{B}\times\mathbf{J} + \mathbf{B}\times(\nabla\times\delta\mathbf{B}) \qquad \mathbf{J}\times\mathbf{B} = \nabla P$$

$$\delta\mathbf{B} = \nabla\times(\xi\times\mathbf{B}) \qquad \nabla\times\mathbf{B} = \mathbf{J}$$

$$0 = \delta p + \xi\cdot\nabla P + \gamma_s P\nabla\cdot\xi. \qquad \nabla\cdot\mathbf{B} = 0.$$

Zero beta plasmas have the following eigenmode equation:

$$\mathbf{B}\cdot\nabla\left(\frac{|\nabla\Phi^*|^2}{B^2}\mathbf{B}\cdot\nabla\xi_s\right) + \rho\omega^2\frac{|\nabla\Phi^*|^2}{B^2}\xi_s = 0.$$

Discrete Alfvén eigenmodes

- Discrete Alfven eigenmodes (AEs) can exist in gaps present in the shear Alfven continuum.
  - Do not experience continuum damping
  - Can be driven unstable by energetic particles (may be important for ITER)
- The toroidicity-induced Alfven eigenmode (TAE) is a well known AE that can exist in toroidal plasmas.
- Analogously, can islands induce a finite gap in the Alfven spectrum?
  - What is the mode structure of the potential instability?

The TAE gap in a nutshell

- A cylindrical geometry exhibits both poloidal and axial directions of symmetry.
- Wrapping the cylinder into a torus results in poloidal variation in the magnetic field strength.
  - This results in the high-field side and low-field side
  - Effectively, the poloidal direction of symmetry is lost
- The loss of symmetry results in a coupling of poloidal mode numbers m_1 and m_2.
- This coupling results in a toroidicity-induced Alfven eigenmode (TAE) gap.
  - Discrete TAE modes can grow unstable in this frequency gap range, unaffected by continuum damping

The TAE gap in a nutshell

Singular solutions to the Alfven eigenmode equation give the continuous spectrum (continuum) of frequencies

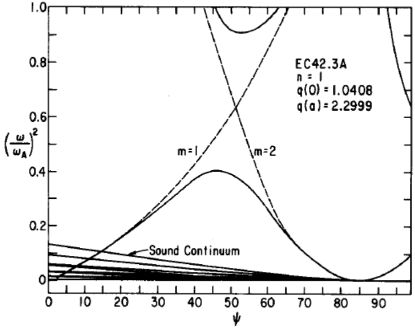
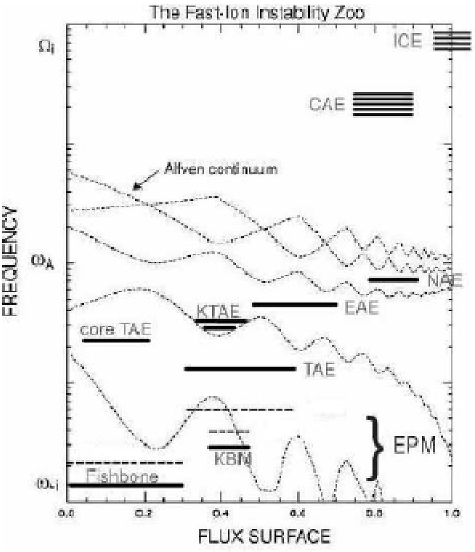


FIG. 1. The shear Alfvén continous spectrum with gaps for a low-β toroidal equilibrium with $n = 1$, $\beta_{av} = 0.04\%$, $R/a = 4$, $q(0) = 1.0408$, and $q(a) = 2.3$. The uncoupled spectra (dotted line) of $m = 1$ and $m = 2$ cross at the $q = 1.5$ surface. The sound continuum is also shown.

From [Cheng and Chance(1986)]

Zoology



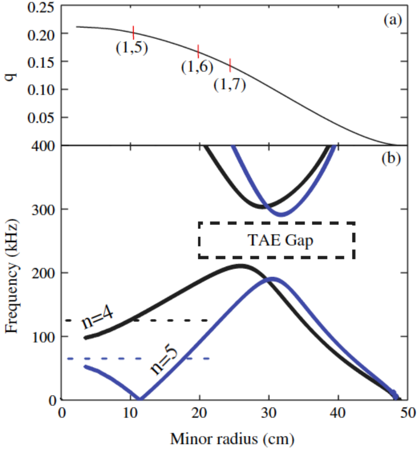
From Biancalani's thesis

An unexplained AE has been observed in MST

- Recently, Jon Koliner observed $n = 4$ and $n = 5$ modes with Alfvenic scaling in 300kA NBI plasmas in MST (see [Koliner et al.(2012)Koliner, Forest, Sarff, and Anderson])
- STELLGAP calculations indicate that the modes are not TAE (toroidicity-induced) modes
- Resonant TAE (rTAE)? KBM? Or something else?...

STELLGAP calculations show AE modes are clearly not TAEs

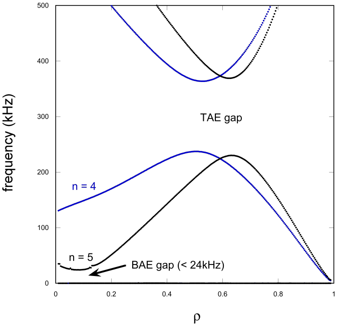
What type of AE mode is this?
[Koliner et al.(2012)Koliner, Forest, Sarff, and Anderson]



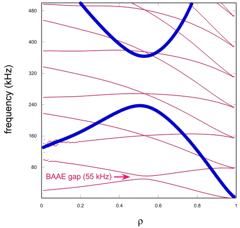
Calculations with compressibility/sound wave effects also seem to show that modes are not BAE/BAAEs

MST: n = 5 BAE gap and n = 4 BAAE gap

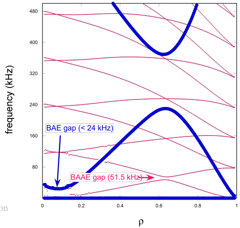
n = 4 and5 Alfvén-acoustic continua
using slow-sound approximation



n = 4 full Alfvén-acoustic continua



n = 5 full Alfvén-acoustic continua



Could an island explain the oberved frequencies of the mode in MST?

- It is known that MST has a large n=5 magnetic island in this configuration
- The island is NOT included in STELLGAP calculations
  - STELLGAP uses a VMEC nested flux surface equilibrium
- The island could have a significant effect on the Alfven continuum
- Can the island help predict the ~60kHz n=5 Alfven activity seen in experiment?

Introduction to magnetic island coordinates

The magnetic field in a cylinder with a magnetic island will be represented using the island helical flux coordinates (Φ^*, χ, α^*) described in [Hegna and Callen(1992)]. These coordinates are defined in terms of the straight field line coordinates (ψ, θ, ζ) of the background cylindrical magnetic field $\mathbf{B_0}$:

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1}$$

$$\mathbf{B_0} = q\nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\psi.$$

The island-producing magnetic field perturbation is assumed of the form $\sqrt{g}\mathbf{B_1} \cdot \nabla\psi = n_0 A \sin(m_0\theta - n_0\zeta - \phi_0)$. This allows a magnetic island to form at the rational surface $q(\psi_0) = q_0 = m_0/n_0$.

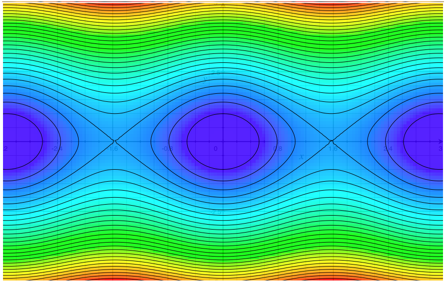
Island magnetic coordinates

In terms of the cylindrical coordinates, the helical coordinates are

$$
\begin{aligned}
x &= \psi - \psi_0 \\
\alpha &= \zeta - q_0 \theta + \frac{\phi_0}{n_0} \\
\chi &= \frac{\theta - \zeta}{1 - q_0} \\
\Psi^* &\approx q_0' \frac{x^2}{2} - A \cos\left(n_0 \alpha\right) \\
k^2 &= \frac{2A}{A + \left|\Psi^*\right|} \\
\Phi^* &= \pm \frac{wE\left(k\right)}{\pi k} \\
\alpha^* &= \frac{\pi}{n_0 K\left(k\right)} F\left(\frac{n_0 \alpha}{2}, k\right).
\end{aligned}
$$

Magnetic island geometry

- Ψ^*/Φ^* are both helical flux surface labels
- α^* is the "poloidal-like" angle
- χ is the angle into the helical island (taken here as a symmetry direction)



For $q_0' > 0$,

- $\Psi^* = -A$ at the O-point
- $\Psi^* = A$ at the separatrix

Straight field-line coordinates in the presence of an island

In the island coordinate system, the total magnetic field can be written in a straight field-line form as

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} = \nabla\alpha^* \times \nabla\Phi^* + \Omega\nabla\Phi^* \times \nabla\chi,$$

where $\Omega = d\Psi^*/d\Phi^*$ is the island rotational transform, given by

$$\Omega(\Psi^*) = \pm\frac{\pi q_0' w}{4kK(k)}$$

outside the island separatrix and

$$\Omega(\Psi^*) = \pm\frac{\pi q_0' w}{8K(\kappa)}$$

$$\kappa^2 = \frac{\Psi^* + A}{2A}$$

inside the separatrix.

This allows us to write $\mathbf{B} \cdot \nabla$ in a much simpler form.

Linearized ideal MHD

The linearized ideal MHD equations are the momentum equation, the combined Faraday's law/Ohm's law, and the equation of state:

$$\rho\omega^2\xi = \nabla\delta p + \delta\mathbf{B}\times\mathbf{J} + \mathbf{B}\times(\nabla\times\delta\mathbf{B})$$
$$\delta\mathbf{B} = \nabla\times(\xi\times\mathbf{B})$$
$$0 = \delta p + \xi\cdot\nabla P + \gamma_s P\nabla\cdot\xi.$$

$$\mathbf{J}\times\mathbf{B} = \nabla P$$
$$\nabla\times\mathbf{B} = \mathbf{J}$$
$$\nabla\cdot\mathbf{B} = 0.$$

Eigenvalue equation

This equation can be simplified for the background cylinder case of interest and by considering zero beta plasmas $P = 0$. Looking for periodic solutions with non-square-integrable radial singularity results in the eigenvalue equation from [Cheng and Chance(1986)]:

$$\mathbf{B}\cdot\nabla\left(\frac{|\nabla\Phi^*|^2}{B^2}\mathbf{B}\cdot\nabla\xi_s\right)+\rho\omega^2\frac{|\nabla\Phi^*|^2}{B^2}\xi_s=0.$$

The surface displacement will be assumed to be of the following form:

$$\xi_s(\chi,\alpha^*)=\xi_0(\alpha^*)\,e^{-il\chi}.$$

Under this assumption, the $\mathbf{B}\cdot\nabla$ operator can be expressed as

$$\mathbf{B}\cdot\nabla\xi_s=\frac{1}{\sqrt{g}}\left(\Omega\frac{\partial}{\partial\alpha^*}-il\right)\xi_s=\frac{1}{\sqrt{g}}e^{\frac{il}{\Omega}\alpha^*}\Omega\frac{\partial}{\partial\alpha^*}\left(\xi_s e^{-\frac{il}{\Omega}\alpha^*}\right).$$

General form of Alfvén eigenmode equation

Thus the differential equation can be written as the second-order ODE:

$$\frac{d}{d\alpha^*}\left(|\nabla\Phi^*|^2\frac{d}{d\alpha^*}Y\right)+\rho\omega^2\frac{\sqrt{g}^2}{\Omega^2}|\nabla\Phi^*|^2 Y=0,$$

$$Y=\xi_0(\alpha^*)\,e^{-\frac{il}{\Omega}\alpha^*}.$$

The structure of the differential equation can be simplified in terms of $x=x(\Psi^*,\alpha^*)$ and α^*. The resulting form looks like

$$\frac{d}{d\alpha*}\left[x^2\frac{d}{d\alpha^*}Y\right]+\frac{\omega^2x^2}{\omega_A^2\Omega^2}Y=0$$

$$x^2=\frac{2}{q_0'}\left(\Psi^*+A\cos\alpha(\alpha^*)\right)$$

$$\omega_A=\frac{2\pi v_A}{q_0 L}$$

This is a Sturm-Liouville problem in α^* for each flux surface Ψ^*.

Solution near the O-point

Under the assumption that $q_0' > 0$, the O-point is located at $\alpha = 0$ and $\Psi^* = -A$. On a flux surface very near the O-point, $|\alpha|$ will be very small everywhere on the surface, taking a maximum value of $\alpha_{max} = \varepsilon << 1$. The radial coordinate x can then be written as

$$x^2 = \frac{2A}{q_0'}(-1 + \cos\varepsilon) \approx \frac{A\varepsilon^2}{q_0'}.$$

This simplifes the differential equation to

$$\frac{d^2 Y}{d\alpha^{*2}} + \frac{\rho\omega^2\sqrt{g}^2}{q_0' A n_0^2 \varepsilon^2} Y \approx 0$$

Solution near the O-point (cont'd)

$$\frac{d^2 Y}{d\alpha^{*2}} + \frac{\rho\omega^2\sqrt{g}^2}{q_0' A n_0^2 \varepsilon^2} Y \approx 0$$

This differential equation describes a simple harmonic oscillator, and periodicity must be enforced:

$$Y = Y_0 e^{i\frac{\omega}{\omega_A \Omega}\alpha^*} = Y_0 e^{ij\alpha^*}$$

This gives us our condition for solution as

$$\frac{\omega}{\omega_A \Omega} = j$$

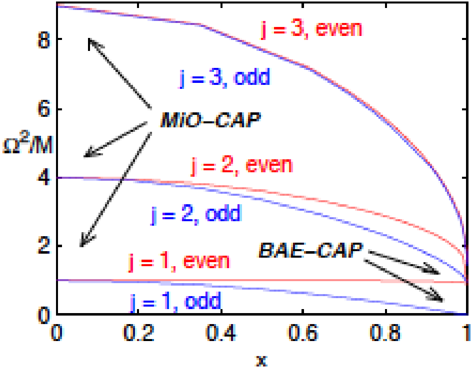
Finally, the Alfven frequencies in the vicinity of the O-point can be found with $\Omega \approx q_0' w/4$:

$$\omega^2 = j^2 \Omega^2 \omega_A^2 = \frac{1}{16} j^2 \left(q_0' w\right)^2 \omega_A^2$$

This agrees with the result from [Biancalani et al.(2011)Biancalani, Chen, Pegoraro, and Zonca].

Computed behavior near O-point agrees with analytical prediction

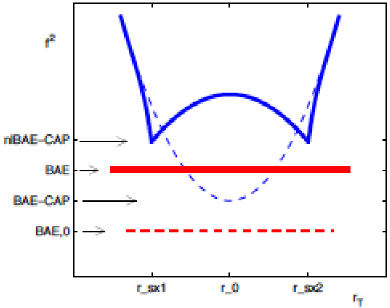
$$\omega^2 = j^2 \Omega^2 \omega_A^2$$

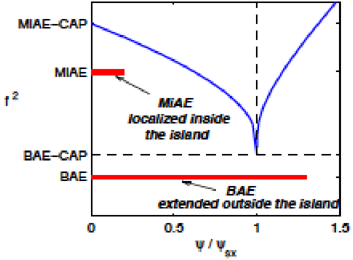


Courtesy: Biancalani's thesis

Biancalani's model describes upshift in BAE frequency

Numerical calculations (including finite beta) show that the spectrum is modified, with a new higher BAE-CAP frequency at the separatrix.





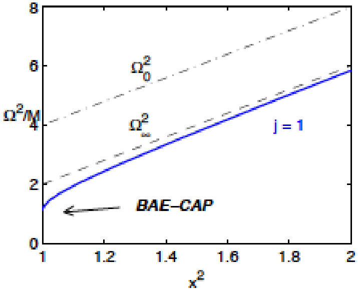
Courtesy: Biancalani's thesis

Physical mechanism for modification

- Field lines on a rational surface close on themselves in analogy to a closed string
- Standing waves – lowest energy state is longest wavelength
- Thus the lowest $k_{//}$ and lowest frequency occur at the rational surface
- When island is present, the separatrix with X-point shortens the length of the "string", leading to a shorter wavelength and raising the frequency.

Numerical solution gives model for CAP at the separatrix

Shooting method code is used to solve eigenmode equations. Solution shows that at separatrix, $\Omega^2_{nlBAE-CAP} = n^2_{isl} M$ ($n_{isl} = 1$ shown here) is the lowest frequency of the spectrum (BAE-CAP).



Courtesy: Biancalani's thesis

With the definitions $M = q_0^2 s^2 w^2 / 4 r_0^2$ and $\Omega^2 = (\omega^2 - \omega^2_{BAE-CAP})/\omega_A^2$, one arrives at the nonlinear BAE-CAP frequency due to the island:

$$f_{nlBAE-CAP} = f_{BAE-CAP} \sqrt{1 + \frac{q_0^2 s^2 n_{isl}^2 w^2 f_A^2}{4 r_0^2 f^2_{BAE-CAP}}}$$

Upshifted BAE frequency due to an island

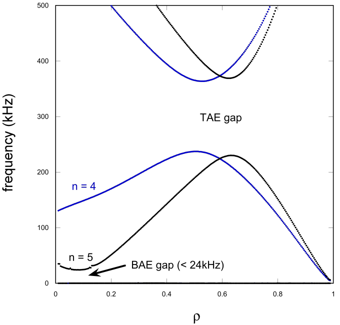
Under the assumption that the BAE frequency is shifted approximately the same as the BAE-CAP frequency, one arrives at the upshift equation for the BAE mode:

$$f_{BAE} = f_{BAE,0}\sqrt{1 + \frac{q_0^2 s^2 n_{isl}^2 w^2 f_A^2}{4 r_0^2 f_{BAE-CAP}^2}}$$

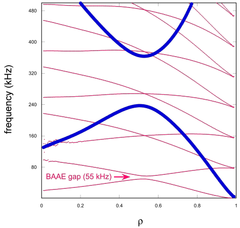
STELLGAP-computed BAE frequency should be upshifted due to the island

MST: n = 5 BAE gap and n = 4 BAAE gap

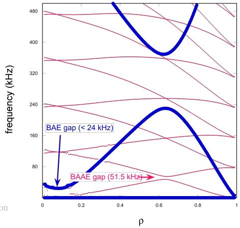
n = 4 and5 Alfvén-acoustic continua using slow-sound approximation



n = 4 full Alfvén-acoustic continua



n = 5 full Alfvén-acoustic continua



The model can explain the observed frequency of the n=5 Alfvenic mode in MST

STELLGAP calculations of the CAP frequency yield a BAE frequency in line with observation

$$
\begin{aligned}
f_{BAE-CAP} &= 24 kHz \\
q_0 &= 0.2 \\
s &= 0.25 \\
n_{isl} &= 5 \\
w &= 10 cm \\
f_A &= 740 kHz \\
r_0 &= 10 cm
\end{aligned}
$$

These values give $f_{nlBAE-CAP} = f_{BAE-CAP}\sqrt{1 + \frac{q_0^2 s^2 n_{isl}^2 w^2 f_A^2}{4 r_0^2 f_{BAE-CAP}^2}} = 96 kHz$.

Zonca's model for the BAE-CAP frequency also gives consistent predictions

$$f_{BAE-CAP} = \frac{1}{2\pi R_0}\sqrt{\frac{2T_i}{m_i}\left(\frac{7}{4}+\frac{T_e}{T_i}\right)} = 32kHz$$

$$q_0 = 0.2$$
$$s = 0.25$$
$$n_{isl} = 5$$
$$w = 10cm$$
$$f_A = 740kHz$$
$$r_0 = 10cm$$

These values give $f_{nlBAE-CAP} = f_{BAE-CAP}\sqrt{1+\frac{q_0^2 s^2 n_{isl}^2 w^2 f_A^2}{4r_0^2 f_{BAE-CAP}^2}} = 98kHz.$

SIESTA can be used to predict the Alfven spectrum in the presence of an island

- The Hessian from SIESTA can be used to solve for the full MHD spectrum for an equilibrium
- The inertia matrix T still needs to be computed to account for the mass density
- The generalized eigenvalue problem looks as follows:

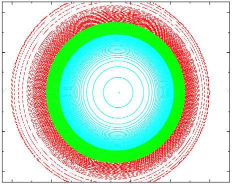
$$H_{ij}\xi^j = -\omega^2 T_{ik}\xi^k$$

$$H = \frac{\partial \mathbf{F}}{\partial \xi} = -\frac{\partial^2 W}{\partial \xi^{\mathbf{2}}}$$

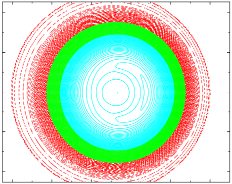
$$T_{ik} = \frac{\rho g_{ik}}{\sqrt{g}}$$

MST SIESTA simulations were performed with varying helical perturbations yielding a range in island size

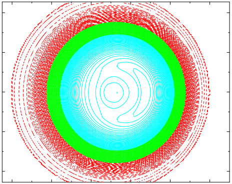
No island



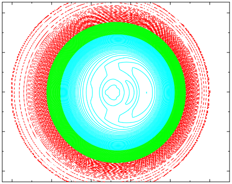
3*10^-3



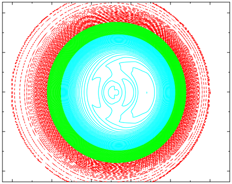
6*10^-3



9*10^-3

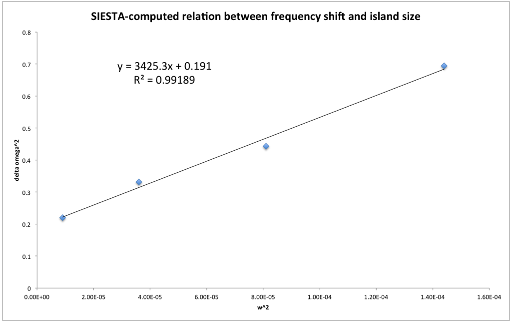


12*10^-3



Initial MST SIESTA simulations confirm scaling of modified BAE frequency with island size

$$\Delta\omega^2_{BAE-CAP} = \frac{n_{isl}^2 f_A^2}{4}\left(\frac{dq}{dr}\right)^2 w^2$$



Summary and future work

- A topologically-toroidal VMEC MST equilibrium does not give STELLGAP predictions in line with experiment
- The magnetic island present in MST may explain the observed Alfven mode frequency
- The inertia matrix must be properly taken into account to compare computation with experimental observations
- A future study of the frequency modification vs. shear will give more confirmation to theory
- This will serve as useful validation for the theory of the Alfven spectrum in the presence of an island and an interesting application for SIESTA

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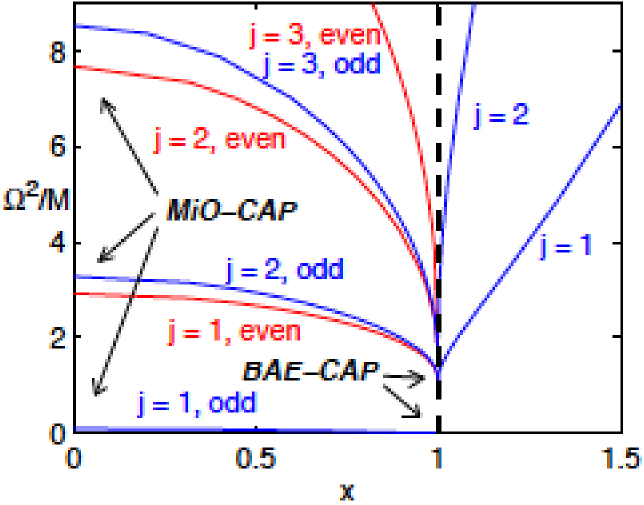
doi: 10.1103/PhysRevLett.109.115003.
URL

Bonus slide: spectrum for large eccentricity case



Solution outside the separatrix

Next let's look at the asymptotic regime far outside the separatrix, that is a flux surface satisfying $A/\Psi^* << 1$. Rewriting the general equation in terms of α and employing a Taylor expansion in our small parameter $A/\Psi*$, we can obtain

$$\left(1+\frac{3A}{2\Psi^*}cosn_0\alpha\right)\frac{d}{d\alpha}\left[\left(1+\frac{3A}{2\Psi^*}cosn_0\alpha\right)\frac{d}{d\alpha}Y\right]$$

$$+\frac{\rho\omega^2\sqrt{g}^2}{2\Psi^* q_0'}\left(1+\frac{2A}{\Psi^*}cosn_0\alpha\right)Y=0.$$

In order to make analytical progress, we make a substitution of variables using the following definition:

$$\frac{d}{dv}=\left(1+\frac{3A}{2\Psi^*}cosn_0\alpha\right)\frac{d}{d\alpha},$$

$$n_0v=n_0\alpha-\frac{3A}{2\Psi^*}sinn_0\alpha.$$

Solution outside the separatrix (cont'd)

The differential equation can now be rewritten in terms of z, using the substitution $n_0 v = 2z$, resulting in a Mathieu's equation:

$$\frac{d^2 Y}{dz^2} + \frac{2\rho\omega^2 \sqrt{g}^2}{q_0' \Psi^* n_0^2} \left[1 + \frac{2A}{\Psi^*} \cos 2z \right] Y = 0.$$

Comparison to the canonical form of Mathieu's differential equation,

$$\frac{d^2 Y}{dz^2} + [a - 2b \cos 2z] \, Y = 0,$$

results in the identifications

$$a = \frac{2\rho\omega^2 \sqrt{g}^2}{q_0' \Psi^* n_0^2},$$

$$b = \frac{-2\rho\omega^2 \sqrt{g}^2}{q_0' \Psi^* n_0^2} \frac{A}{\Psi^*}.$$

Solution outside the separatrix (cont'd)

For a given nonzero value of b, the Mathieu functions that satisfy the differential equation are only periodic in z for certain values of a corresponding to an integer k. The lowest Mathieu characteristic pair $(k = 1)$ defines the island-induced Alfvén gap. The functional form for $a(b)$ for the two characteristics of $k = 1$ with $|b| << 1$ and $b < 0$ is given by

$$a(b) = 1 \pm b.$$

Substituting the values for a and b, one obtains

$$\frac{2\rho\omega^2\sqrt{g}^2}{q_0'\Psi^* n_0^2} = 1 \pm \frac{2\rho\omega^2\sqrt{g}^2}{q_0'\Psi^* n_0^2}\frac{A}{\Psi^*}.$$

Solution outside the separatrix (cont'd)

Thus to zeroth order, we have $\omega_0^2 = q_0' n_0^2 \Psi^* / 2\rho \sqrt{g}^2$. Using this and working to first order in A/Ψ^* results in the final expression for the island-induced Alfvén eigenmode (IAE) gap in the frequency spectrum far outside the separatrix:

$$\omega_\pm^2 = \frac{|q_0'| \, \Psi^* n_0^2}{2\mu_0 \rho \sqrt{g}^2} \left[1 \pm \frac{A}{\Psi^*}\right] \sim \frac{1}{2} \left|q_0'\right| \Psi^* v_A^2 k_\parallel^2 [1 \pm \frac{A}{\Psi^*}].$$