Analysis of the eigenspectrum of the MHD force operator in the SIESTA equilibrium code

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Overview

o SIESTA (Scalable Iterative Equilibrium Solver for Toroidal Applications) is a 3D nonlinear ideal MHD equilibrium solver capable of resolving islands in confinement devices in an accurate and scalable manner.

o The presence of a numerical nullspace of the Hessian matrix has important convergence implications for SIESTA.

o The structure of the nullspace eigenmodes has been calculated and compares favorably with expectations. The calculations were done on a field-period 3 stellarator.

o A detailed stability analysis has been carried out for a CDX-U tokamak configuration that is Mercier unstable.

o There is a decrease in unstable modes from the VMEC axisymmetric equilibrium to the SIESTA equilibrium containing an m=1, n=1 island.

The Hessian matrix

Total energy for a stationary plasma (magnetic + internal):

$$W = \int \left[\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} \right] dVol$$

The Hessian matrix used in SIESTA is defined below. The eigenspectrum for this matrix will be discussed throughout the poster for both VMEC and SIESTA equilibria.

For quadratic energies, the Hessian is essentially the coefficient of the second-order term (with a negative sign due to the conventions used in the code). For a spring potential, the Hessian would essentially be the spring constant.

$$W = \frac{1}{2} k \vec{\xi}^2$$

$$\vec{H} = -\frac{\partial^2 W}{\partial \vec{\xi}^2} = -k\vec{I}$$

$$\vec{H} = \frac{\partial \vec{F}}{\partial \vec{\xi}} = -\frac{\partial^2 W}{\partial \vec{\xi}^2}$$

This Hessian is negative definite (has only negative eigenvalues) for a completely stable equilibrium.

The analytical nullspace

The nullspace of the Hessian matrix in SIESTA is very important because it can lead to huge displacements in directions that result in no change to the MHD force. It can easily be seen from the linearized ideal MHD equations that a plasma displacement that is purely parallel to the magnetic field everywhere will result in zero contribution to the linear force.

$$\delta \vec{F} = \delta \vec{J} \times \vec{B}_0 + \vec{J}_0 \times \delta \vec{B} - \nabla \delta p$$

Here incompressibility is used, ie

$$\delta \vec{B} = \nabla \times \left(\vec{\xi}_{//} \times \vec{B}_0 \right) = 0$$

$$\nabla \cdot \vec{\xi} = 0$$

$$\delta \vec{J} = \frac{1}{\mu_0} \left(\nabla \times \delta \vec{B} \right) = 0$$

$$\delta p = (\gamma - 1)\vec{\xi}_{//} \cdot \nabla p_0 - \gamma \nabla \cdot \left(p_0 \vec{\xi}_{//} \right) = -\gamma p_0 \nabla \cdot \vec{\xi}_{//} = 0$$

$$\delta \vec{F} = \delta \vec{J} \times \vec{B}_0 + \vec{J}_0 \times \delta \vec{B} - \nabla \delta p = 0$$

Thus we would expect that a parallel plasma displacement would serve as a nullspace vector for the Hessian matrix; that is a parallel displacement should satisfy the following equation:
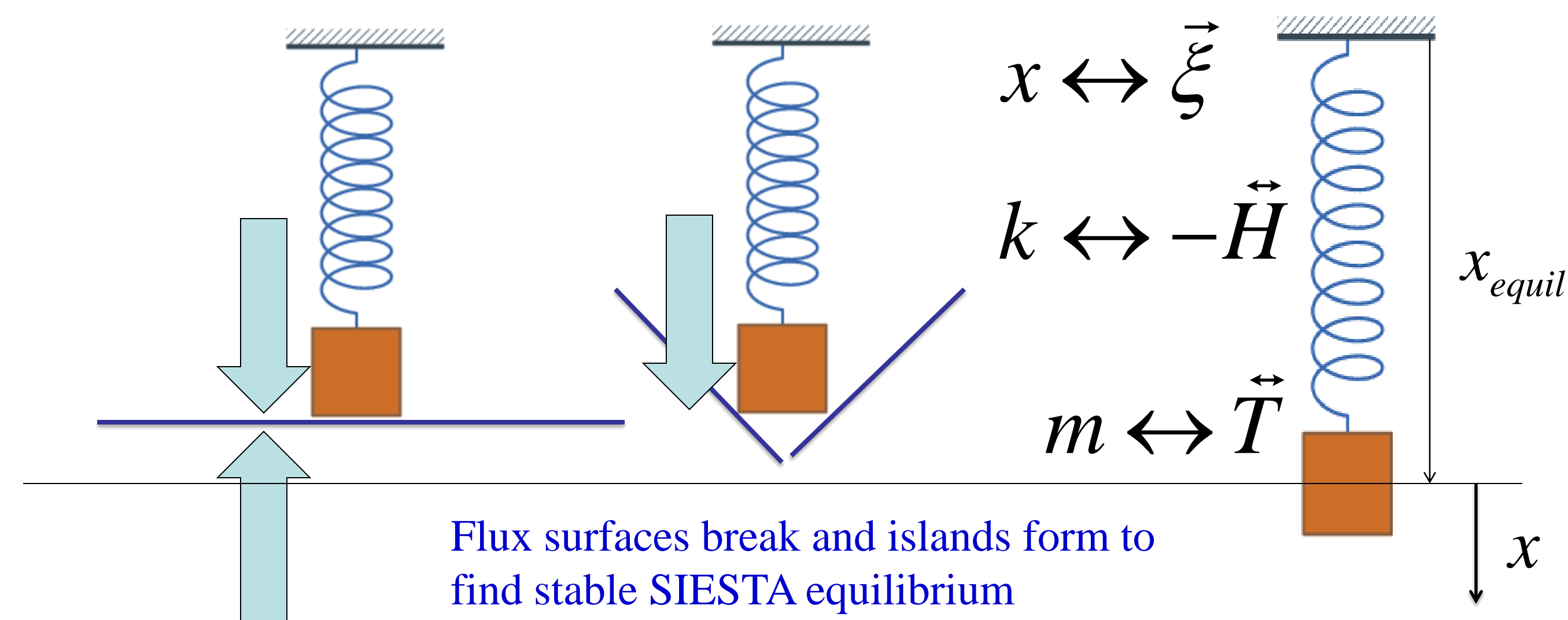
$$\vec{H}\vec{\xi}_{//} = \frac{\partial \vec{F}}{\partial \vec{\xi}} \vec{\xi}_{//} = \delta \vec{F} = 0$$

Put another way, we hope to find numerically that the nullspace eigenvectors of the SIESTA Hessian matrix are displacements that are essentially parallel to the magnetic field everywhere in the domain. This has been verified as discussed to the right.

Mechanical analogue for eigenproblem

VMEC equilibrium is ideal MHD stable

SIESTA includes resistivity and non-ideal perturbations to allow for tearing mode instabilities

$$x \leftrightarrow \vec{\xi}$$

$$k \leftrightarrow -\vec{H}$$

$$m \leftrightarrow \vec{T}$$

Flux surfaces break and islands form to find stable SIESTA equilibrium

Spring

A normal mode analysis on the equations of motion gives the following expression:

$$x \sim e^{i\omega t}$$

$$-kx = -\omega^2 m x$$

Leading to the normal mode eigenfrequency:

$$\omega^2 = \frac{k}{m}$$

MHD

The normal mode analysis for ideal MHD proceeds in a similar fashion

$$\vec{\xi} \sim e^{i\omega t}$$

$$\vec{H}\vec{\xi} = -\omega^2 \vec{T} \vec{\xi}$$

The following equation gives the MHD eigenfrequncies:

$$\det | \vec{H} + \omega^2 \vec{T} | = 0$$

## Nullspace structure for l=3 stellarator

Numerically, we are worried about the eigenproblem in which the inertia (T) matrix is set to unity: In this case the eigenvalue problem is given by the equation to the right.
We want to solve for the eigenfrequencies and eigenmodes.
Note that these are not the MHD normal modes as we have set the inertia matrix to the identity matrix.

$$\vec{T} = \vec{I}$$

$$\vec{H}\vec{\xi} = -\omega^2 \vec{\xi}$$

Real eigenfrequencies correspond to numerically stable modes while imaginary eigenfrequencies correspond to numerically unstable modes. For a classical, l=3 stellarator, a nullspace eigenvector was plotted below. This plasma displacement would result in essentially no change in the linearized MHD force.
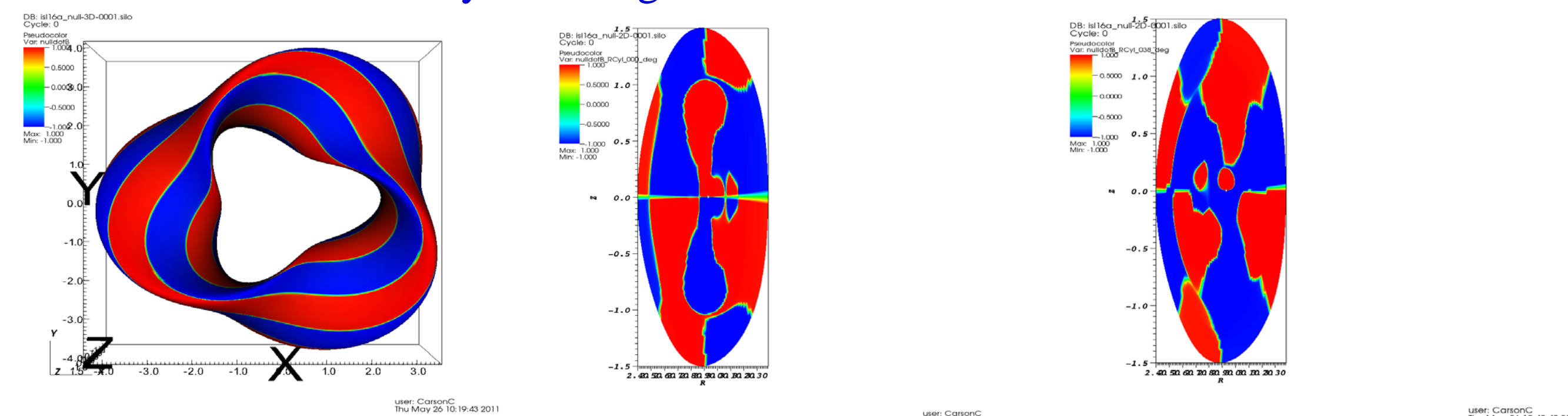
These plots of the nullvector dotted into the magnetic field show that the mode is parallel (red) or antiparallel (blue) to the magnetic field everywhere, as expected. There is a large degeneracy of eigenmodes resulting in no net force.
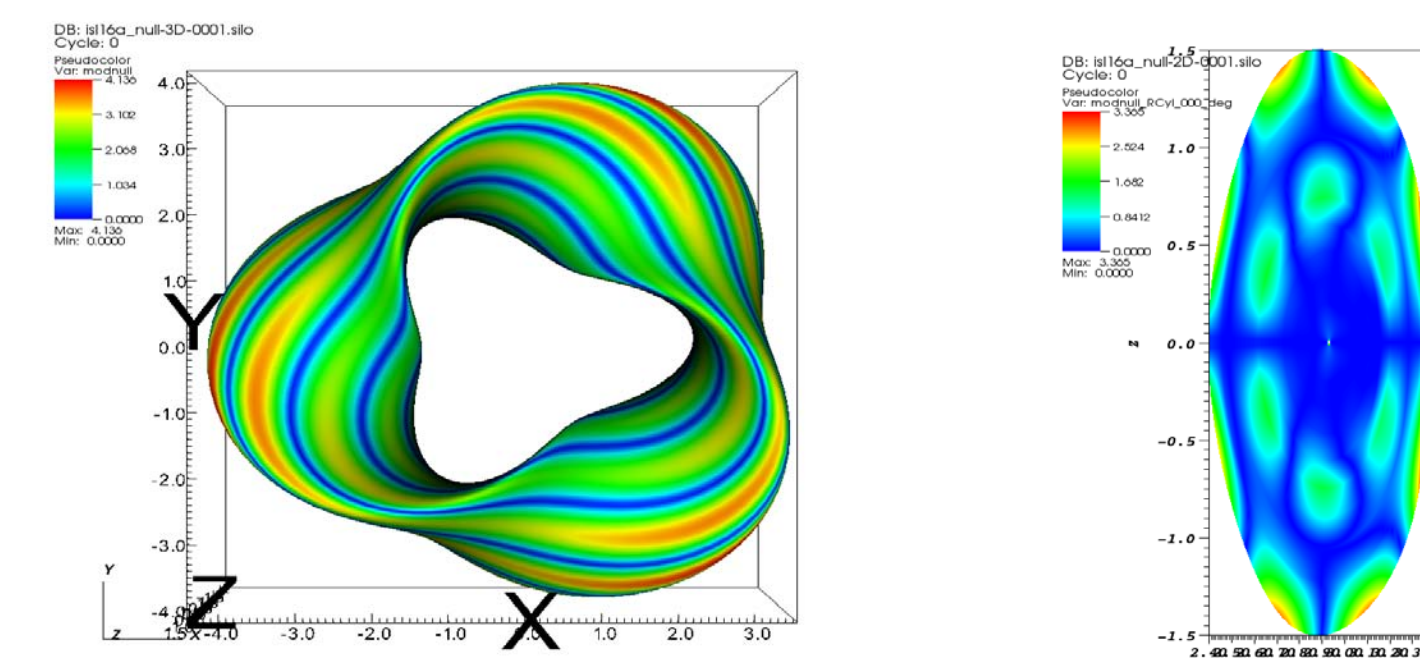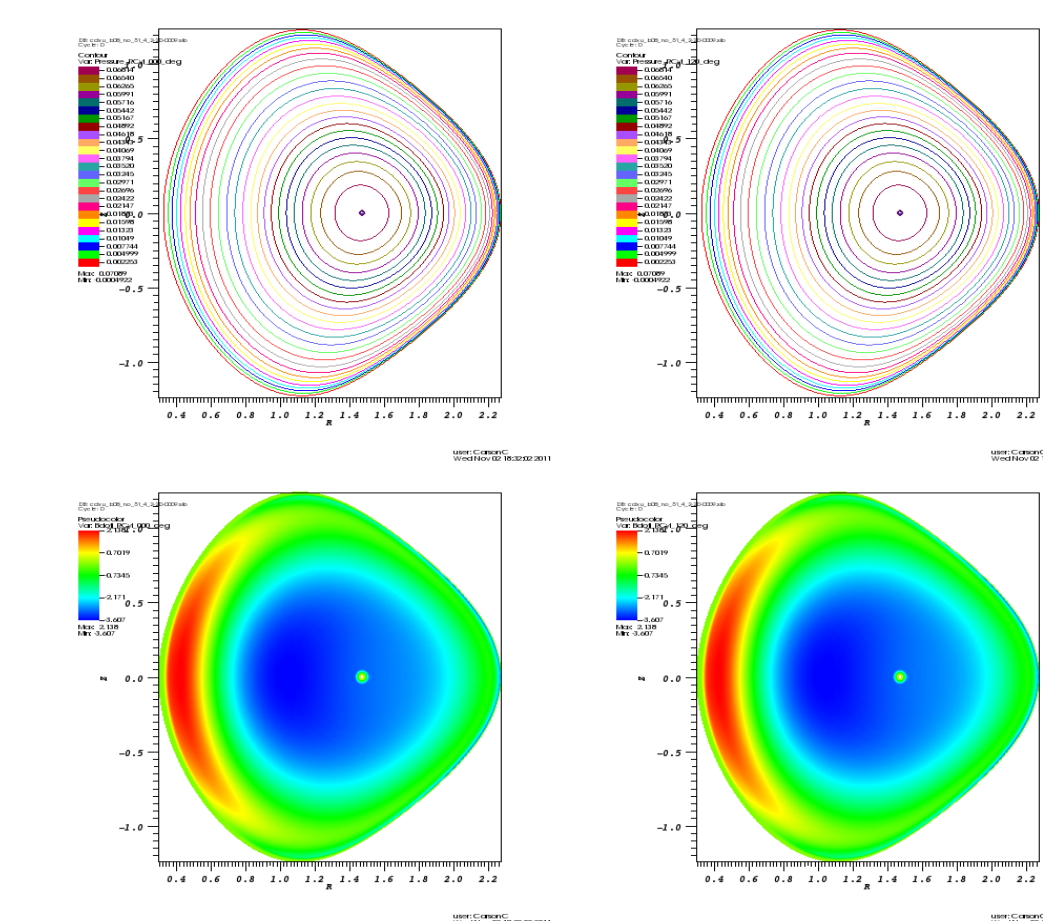
Plots of modulus of the nullspace eigenmode

High beta CDX-U stability analysis

A stability analysis was performed on a high beta (8%) CDX-U equilibrium that is known to be Mercier unstable. CDX-U (Current Drive Experiment-Upgrade) is a small tokamak located at Princeton Plasma Physics Lab.
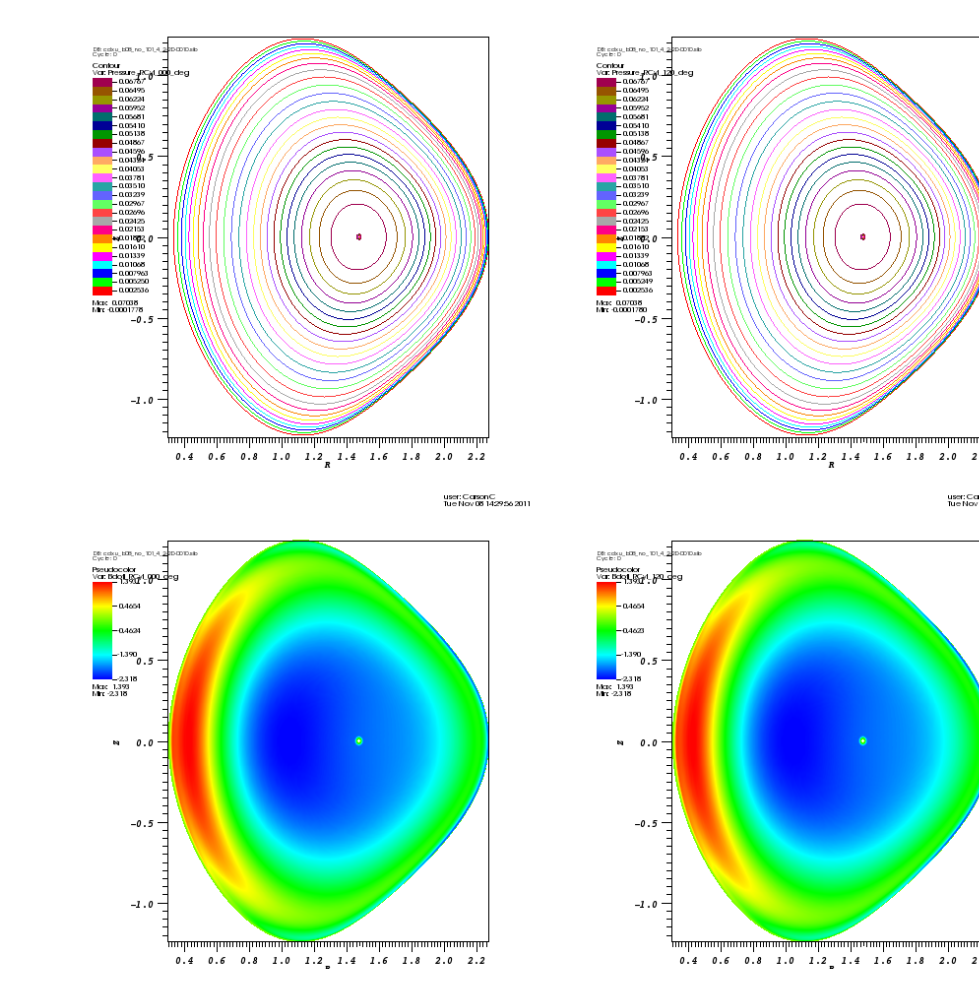
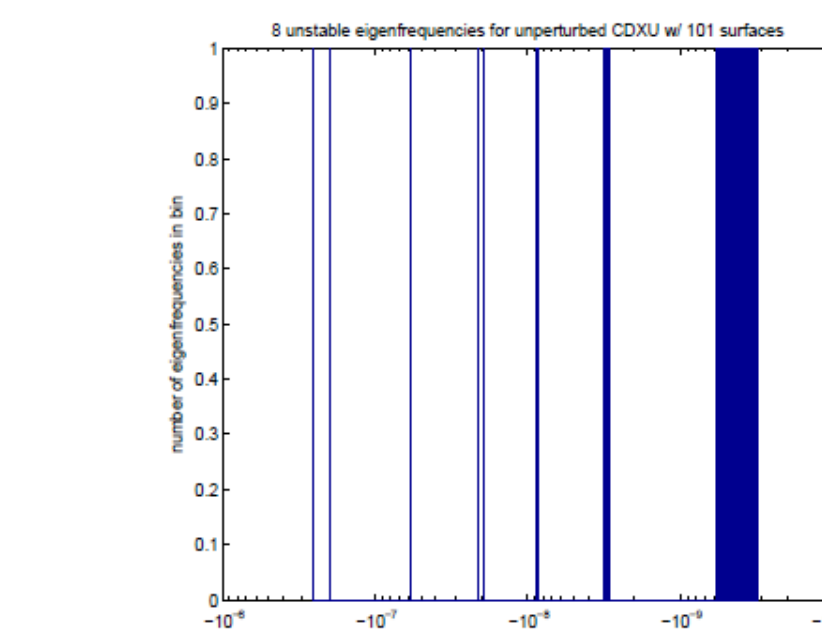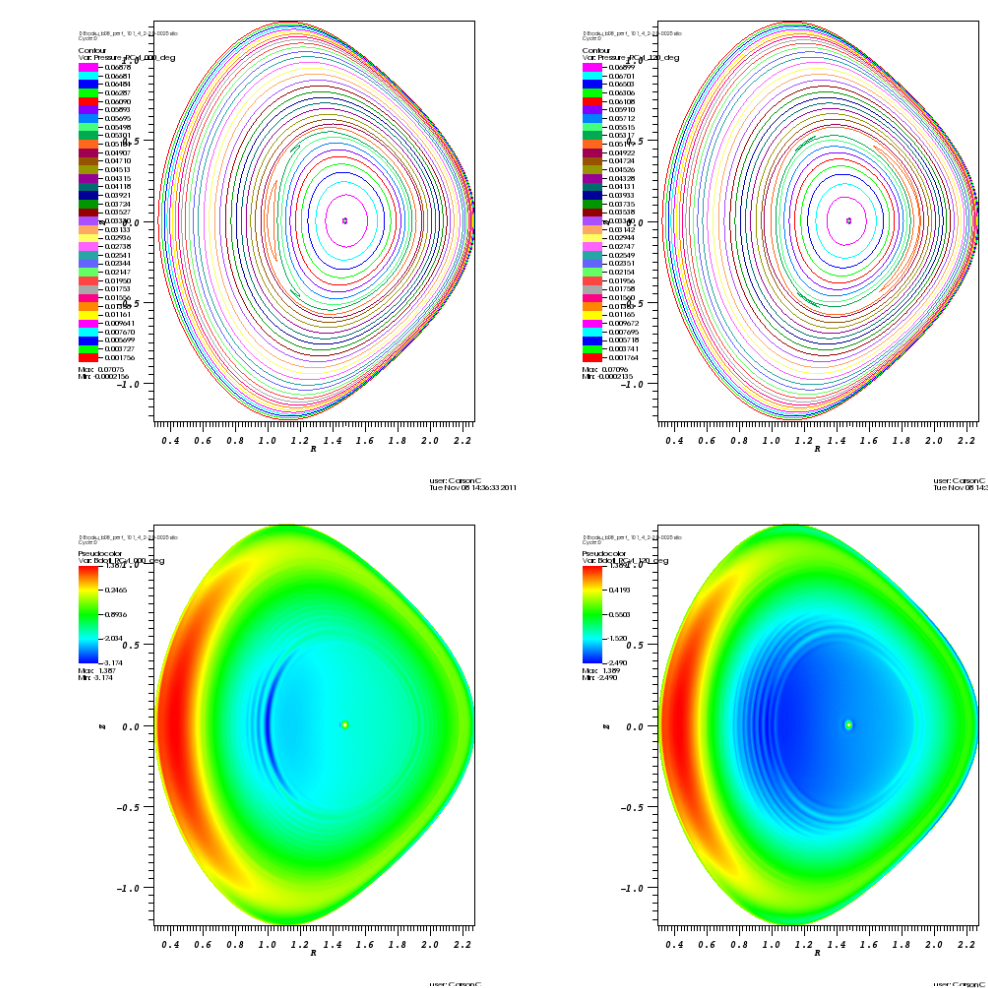CDXU axisymmetric VMEC equilibrium converged on SIESTA mesh (51 surfaces)

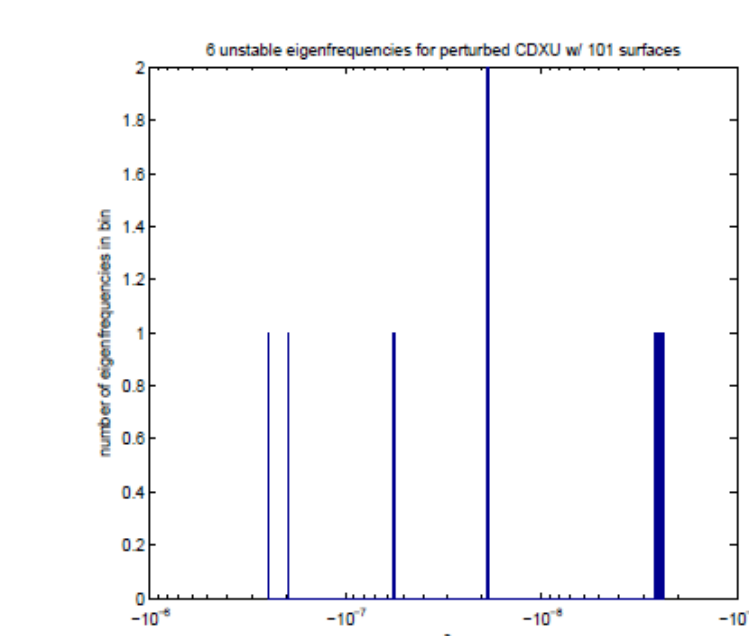CDXU SIESTA equilibrium w/ m=1, n=1 island (51 surfaces)
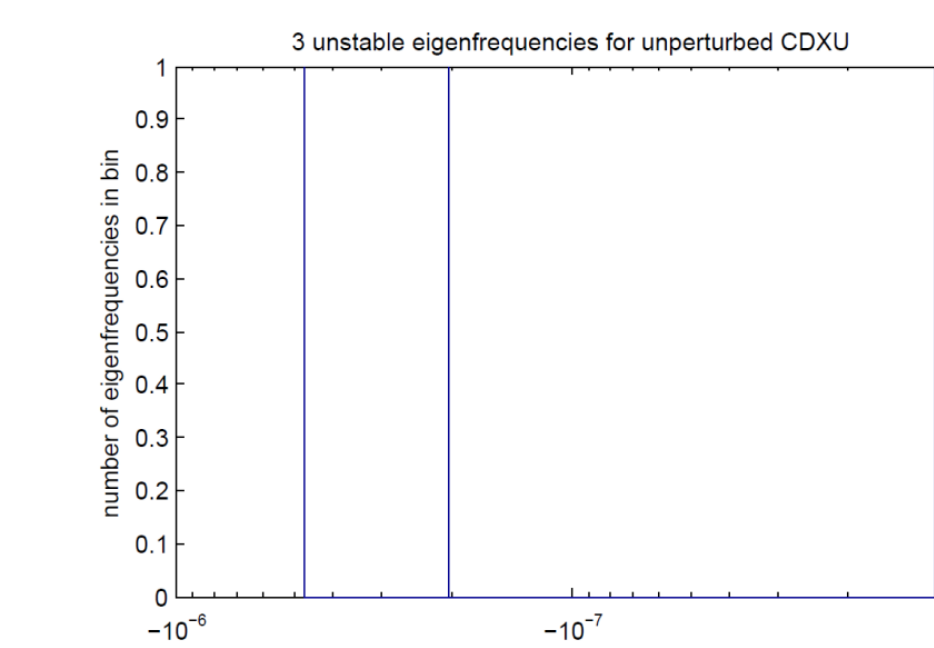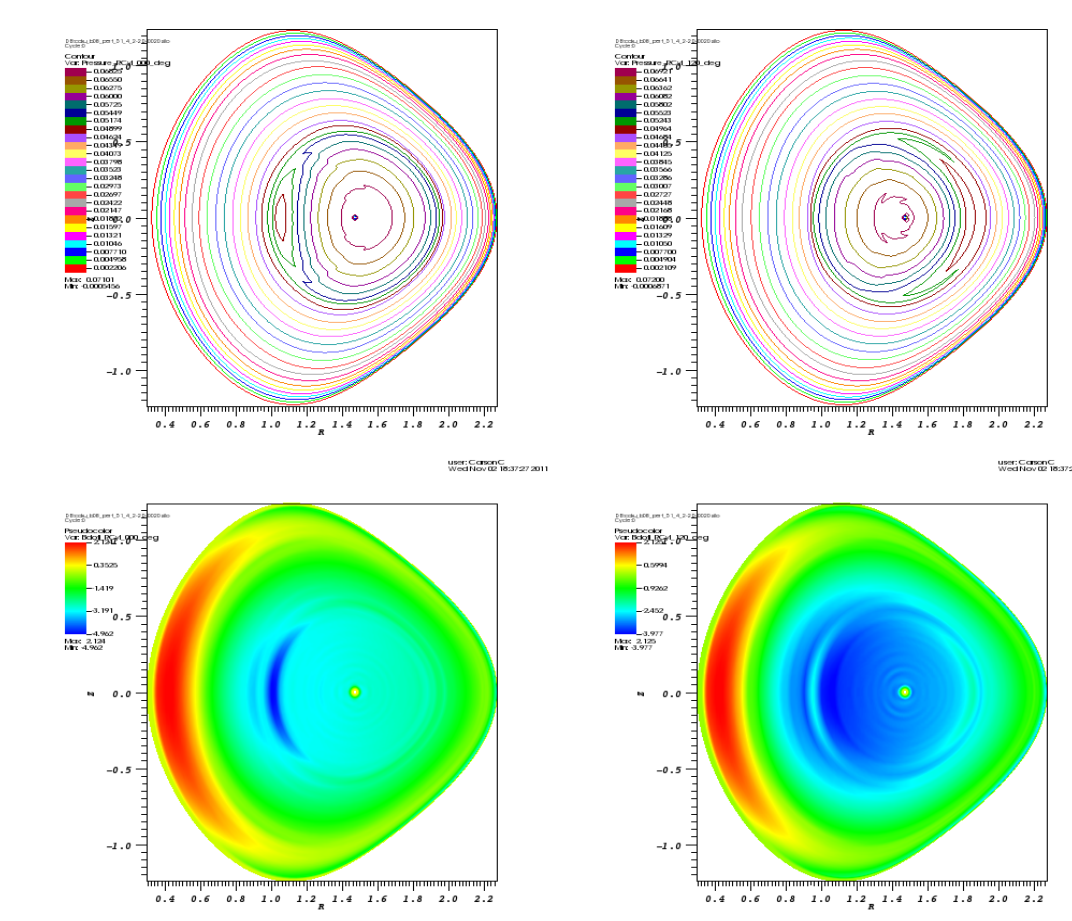
CDXU axisymmetric VMEC equilibrium converged on SIESTA mesh (101 surfaces)

CDXU SIESTA equilibrium w/ m=1, n=1 island (101 surfaces)

The origin and consequences of these remaining unstable modes are under investigation.

Summary

• The nullspace eigenmodes of the SIESTA Hessian matrix have successfully been demonstrated to be parallel plasma displacements, agreeing with theory.
• A stability analysis has been performed on CDX-U, demonstrating the presence of unstable modes in an axisymmetric VMEC equilibrium. After convergence in SIESTA and formation of a magnetic island, the number of unstable modes decreases.
• The remaining instabilities are currently being considered. They could be due to numerical inaccuracies.

References

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